MATH 8000 HOMEWORK 1

DUE ON THURSDAY, AUGUST 24

(1) Show that any finite group of even order contains an element of order two.

(2) Explicitly compute the centralizer of $(1\,2)$ in S_3.

(3) Show that if σ is any permutation of n elements, and $(i_1\,i_2\,\cdots\,i_r)$ is a cycle, then

$$\sigma(i_1\,i_2\,\cdots\,i_r)\sigma^{-1} = (\sigma(i_1)\,\sigma(i_2)\,\cdots\,\sigma(i_r)).$$

Use this result to describe the centralizer of $(1\,2)$ in S_n for any n.

(4) Describe the centralizer of the element

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in $GL_2(\mathbb{R})$.

(5) Let $G = \{(a,b) \in \mathbb{R}^2 \mid a \neq 0\}$. Define a binary operation on G as follows:

$$(a,b)\cdot(c,d) = (ac, ad+b).$$

(a) Show that G is a group under this operation.
(b) Exhibit an injective homomorphism from G to $GL_2(\mathbb{R})$.
(c) Check that $H = \{(1,b) \mid b \in \mathbb{R}\}$ is a normal subgroup of G (that is, show that $ghg^{-1} \in H$ for any $h \in H$ and $g \in G$).
(d) Under your homomorphism from part (b), is the image of H a normal subgroup of $GL_2(\mathbb{R})$?

(6) Show that for any $n \geq 3$, the group A_n is generated by the 3-cycles in S_n. In other words, show that any even permutation is a product of 3-cycles.

(7) Consider the multiplicative group $(\mathbb{Z}/n)^\times$, consisting of all integers modulo n that have a multiplicative inverse modulo n.
(a) Check that $(\mathbb{Z}/7)^\times$ is cyclic.
(b) Give an example to show that this group is not cyclic in general.

(8) A group G is called *finitely generated* if there is a finite subset $A \subset G$ such that G is generated by A, that is, $\langle A \rangle = G$.
(a) Show that $(\mathbb{Z}^i, +, 0)$ is finitely generated.
(b) Show that $(\mathbb{Q}, +, 0)$ is not finitely generated.