MATH 8000 HOMEWORK 10

Due on Thursday, November 16

(1) Prove that the Galois group of $x^p - 2$, where p is a prime, is isomorphic to the following subgroup of $GL_2(\mathbb{F}_p)$:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F}_p,\ a \neq 0 \right\}.$$

(2) Let $f(x) \in \mathbb{Q}[x]$ be a monic irreducible quartic polynomial whose Galois group is isomorphic to S_4 over $\mathbb{Q}$. If θ is a root of $f(x)$, show that there is no proper intermediate field $\mathbb{Q} \subsetneq F \subsetneq \mathbb{Q}(\theta)$. Is $\mathbb{Q}(\theta)/\mathbb{Q}$ a Galois extension?

(3) Let E be a splitting field of $x^6 - 3$ over $\mathbb{Q}$. Let $\alpha = 3^{1/6}$.
  (a) Show that $E = \mathbb{Q}(\alpha, i)$, and hence show that $[E : \mathbb{Q}] = 12$.
  (b) Is the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ abelian?
  (c) Find $\mathrm{Gal}(E/\mathbb{Q})$, including a presentation by generators and relations. Draw the subfield lattice of $E/\mathbb{Q}$ and the corresponding subgroup lattice of $\mathrm{Gal}(E/\mathbb{Q})$.

(4) Let E be a splitting field of $x^8 - 2$ over $\mathbb{Q}$. Compute everything you can about $\mathrm{Gal}(E/\mathbb{Q})$ (equivalently, about the subfields of E over $\mathbb{Q}$).