MATH 8000 HOMEWORK 2

Due on Thursday, August 31

(1) Describe all subgroups of the additive group $(\mathbb{Z}, +, 0)$.

(2) Consider the subgroup $(\mathbb{Z}, +, 0)$ of $(\mathbb{Q}, +, 0)$. Let p and q be two distinct prime numbers. Show that the (left or right) cosets

$$\frac{1}{p} + \mathbb{Z} \text{ and } \frac{1}{q} + \mathbb{Z}$$

are distinct. Conclude that there are infinitely many $\mathbb{Z}$-cosets in $\mathbb{Q}$. (In other words, the *index* $|\mathbb{Q} : \mathbb{Z}|$ is infinite.)

(3) Let $H < G$. Suppose that $a, b \in G$ such that $Ha = bH$. Show that

$$Ha = aH = bH = Hb.$$

(4) Prove that if p is a prime, then any group of order p^2 is abelian.

(5) Let p be a prime, and let $G = GL_2(\mathbb{F}_p)$ (the invertible 2×2 matrices with entries the integers modulo p).
   (a) Find the order of G. (Hint: a matrix is in GL_2 if and only if its columns are linearly independent.)
   (b) Let a, b be fixed elements of $\mathbb{F}_p$ where $a \neq b$. Find the size of the conjugacy class of the element $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in G.
   (c) Let a be a fixed element of $\mathbb{F}_p$. Find the sizes of the conjugacy classes of the element $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and the element $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ in G.
   (d) Show that these do not cover all the elements of G. (Bonus: Can you say something about which conjugacy classes are missing?)

(6) If G is any group, then its *commutator subgroup* is the subgroup

$$[G, G] := \langle aba^{-1}b^{-1} \mid a, b \in G \rangle.$$

   (a) Show that $[G, G]$ is a normal subgroup of G.
   (b) Show that $G/[G, G]$ is abelian. This is called the *abelianization* of G.
   (c) Find the commutator subgroup of S_4.
   (d) Can you find the commutator subgroup of S_n for any n?