MATH 8000 HOMEWORK 3

Due on Thursday, September 7

(1) Let D_{2n} be the dihedral group of order $2n$ — this is the symmetry group of a rigid regular n-sided polygon, which contains n rotations (including the trivial rotation) and n reflections.

Show that D_{2n} is isomorphic to a quotient of the free group on the set $\{\sigma, \tau\}$ by the subgroup generated by the relations $\{\tau^2 = 1, \sigma^n = 1, \tau\sigma = \sigma^{-1}\tau\}$. In other words, show that

$$D_{2n} \cong \langle \sigma, \tau \mid \tau^2, \sigma^n, \tau\sigma\tau^{-1}\sigma \rangle.$$

(2) Show that if G has a subgroup of index 2, then it is normal.

(3) Let H be a subgroup of G. Let $S = \{gH \mid g \in G\}$ be the set of left cosets of H in G. Consider the action of G on S by left multiplication.

(a) Show that if $\varphi\colon G \to \operatorname{Sym} S$ is the corresponding group homomorphism, and $K = \ker\varphi$, then $K < H$.

(b) Show that $H \lhd G$ if and only if $K = H$.

(c) Show that if p is the smallest prime dividing $|G|$ and $[G : H] = p$, then H is normal. (Hint: try to show that $[G\colon K] = p$.)

(4) Recall that a group G is *simple* if it has no non-trivial normal subgroups. Show that a group of order 284 cannot be simple.

(5) (a) Show that if $H \lhd G$, then H is a union of conjugacy classes.

(b) Find the orders of all conjugacy classes in S_5.

(c) Use the previous parts to show that the only normal subgroups of S_5 are $\{e\}$, A_5, and S_5.

(6) (Fall 2014 qualifying exam) Let G be a group of order 96.

(a) Show that G has either one or three Sylow 2-subgroups.

(b) Show that G either has a normal subgroup of order 32 or a normal subgroup of order 16.