MATH 8000 HOMEWORK 4

Due on Thursday, September 14

(1) An element of order two is called an involution. Show that if $u, v \in G$ are two distinct involutions in a finite group G, then the subgroup generated by u and v is isomorphic to a dihedral group.

(2) For each $i = 1, 2, \dots, n$, let G_i be a group and H_i be a subgroup of G_i. Let $H = \prod_{i=1}^n H_i$ and let $G = \prod_{i=1}^n G_i$.
   (a) Prove that $H \lhd G$ if and only if $H_i \lhd G_i$ for each i.
   (b) If $H \lhd G$, then show that $G/H \cong \prod_{i=1}^n G_i/H_i$.

(3) Let $1 \to H \xrightarrow{\varphi} G \xrightarrow{\psi} K \to 1$ be a short exact sequence of groups. A homomorphism $\sigma \colon K \to G$ is called a *splitting* if $\psi \circ \sigma = \mathrm{id}_K$. (If such a homomorphism exists, then the short exact sequence is said to *split*.)
   (a) Show that if the above short exact sequence splits, then G is isomorphic to a semidirect product of H and K. Explicitly write down the corresponding homomorphism $\alpha \colon K \to \mathrm{Aut}(H)$.
   (b) Show that for any $m, n \in \mathbb{N}$, there is a short exact sequence

$$1 \to \mathbb{Z}/m \to \mathbb{Z}/mn \to \mathbb{Z}/n \to 1,$$

   which splits if and only if $\gcd(m, n) = 1$.

(4) Recall that $SL_n(\mathbb{C}) = \{M \in GL_n(\mathbb{C}) \mid \det(M) = 1\}$.
   (a) Show that there is a short exact sequence

$$1 \to SL_n(\mathbb{C}) \to GL_n(\mathbb{C}) \to \mathbb{C}^\times \to 1,$$

   where $\mathbb{C}^\times$ is the multiplicative group of complex numbers.
   (b) Show that this short exact sequence is split.

(5) Let $K = \mathbb{Z}/7$ be the cyclic group of order 7, generated by $x \in K$.
   (a) Show that K has an automorphism of order 3. (That is, if composed three times, it gives the identity automorphism.)
   (b) Use the previous part to construct (with proof) a nonabelian group of order 21.

(6) Suppose we have a diagram as follows, where the two horizontal lines are short exact sequences, and each square commutes.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & K & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle a} & & \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle c} & & \big\downarrow \\ 1 & \longrightarrow & H' & \xrightarrow{\varphi'} & G' & \xrightarrow{\psi'} & K' & \longrightarrow & 1 \end{array}$$

Show that if a and c are both isomorphisms, then so is b.