MATH 8000 HOMEWORK 5

Due on Thursday, September 28

(1) Let G be a finite group and let p be a prime. Let S_p be the set of P-Sylow subgroups of G. Show that the intersection

$$\bigcap_{P \in S_p} N_G(P)$$

is normal in G.

(2) (Warm-up, not to be turned in.) The *tetrahedral group* T is the group of rotational symmetries of a regular tetrahedron. Find an explicit isomorphism from T to A_4.

(3) The *icosahedral group* I is the group of rotational symmetries of a regular icosahedron. It is also the group of rotational symmetries of a regular dodecahedron — you can inscribe a regular dodecahedron inside a regular icosahedron to get the same symmetry group.

For this problem, you may want to acquire or construct a model of a dodecahedron or icosahedron. I find that a dodecahedron is easier to think about, but YMMV.

(a) Describe all the elements of I and show that $|I| = 60$. (Hint: The axis of rotation has to pass through either a vertex, the interior of an edge, or the interior of a face.)

(b) Show that I has six Sylow 5-subgroups.

(c) Show that I has ten Sylow 3-subgroups.

(d) Show that I has five Sylow 2-subgroups.

(e) Let S be the set of Sylow 2-subgroups of I. Show that for any $P \in S$, the stabilizer of P under conjugation by G (which is exactly $N_I(P)$) has 12 elements.

(f) Show that for any distinct $P_1, P_2 \in S$, the intersection $P_1 \cap P_2$ is trivial.

(g) Show that any non-identity element of $N_I(P) \backslash P$ is a 3-cycle (i.e., has order 3).

(h) Conclude that

$$K = \bigcap_{P \in S} N_I(P)$$

must be trivial. (Hint: What can the order of K be?)

(i) Conclude that the map $\alpha \colon I \to \operatorname{Sym} S$ is injective, and hence conclude that $I \cong A_5$.