MATH 8000 HOMEWORK 6

DUE ON THURSDAY, OCTOBER 5

(1) Let F be a field and let R be any non-zero ring. Show that any homomorphism $\varphi\colon F \to R$ is injective.

(2) Show that if a and b are two elements of a ring R, then $1-ab$ is invertible if and only if $1-ba$ is invertible. (Warning: a and b need not be invertible!)

(3) Classify all ideals in $M_n(\mathbb{C})$.

(4) Let I be an ideal in a commutative ring R.
  (a) Show that $M_n(I)$ is an ideal of $M_n(R)$.
  (b) Prove that
$$M_n(R)/M_n(I) \cong M_n(R/I).$$

(5) (Chinese remainder theorem for rings). We say that $a \equiv b$ modulo an ideal I if $a+I = b+I$, or equivalently, $a-b \in I$.
  Let I and J be ideals of R that are *relatively prime* (that is, $I+J=R$).
  (a) Show that if $a,b \in R$, then there is some $r \in R$ such that $r \equiv a$ modulo I and $r \equiv b$ modulo J.
  (b) Show that $R/(I \cap J) \cong R/I \times R/J$.

(6) Let R be the set of real-valued continuous functions on the interval $[0,1]$. Given $f,g \in R$, we define $(f+g)(x) = f(x)+g(x)$ and $(fg)(x) = f(x)g(x)$ as usual. Then R is a ring under these operations. For the following problem, you may freely use results about the topology of $\mathbb{R}$.
  (a) Let $I \subset R$ be an ideal. Define $V(I) = \{p \in [0,1] \mid f(p) = 0 \text{ for every } f \in I\}$. Show that I is not the unit ideal if and only if $V(I) \neq \emptyset$.
  (b) A proper ideal of a ring is called *maximal* if there is no larger proper ideal containing it. Classify the maximal ideals of R.
  (c) Show that $V(I)$ is a closed subset of $[0,1]$.
  (d) For any closed subset $A \subset [0,1]$, define $I(A)$ as follows:
$$I(A) = \{f \in R \mid f(a) = 0 \text{ for every } a \in A\}.$$
  Show that I is an ideal.
  (e) Show that $V(I(A)) = A$ for any closed subset $A \subset [0,1]$.
  (f) Give a counterexample to show that $I(V(I)) \neq I$.
  (g) (Not to be turned in.) Show that $f \in R$ is a zerodivisor if and only if $V(f) = \{p \in [0,1] \mid f(p) = 0\}$ contains an open interval.