MATH 8000 HOMEWORK 7

Due on Thursday, October 12

(1) Consider $\mathbb{Q}$ as a subring of $\mathbb{R}$.
 (a) Show that $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$ and that the real numbers 1, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$ are linearly independent over $\mathbb{Q}$.
 (b) Show that $u = \sqrt{2} + \sqrt{3}$ is algebraic over $\mathbb{Q}$ and find an ideal I such that $\mathbb{Q}[x]/I \cong \mathbb{Q}[u]$.

(2) Let $\mathbb{R}[\![x]\!]$ be the set of all sequences $\{(a_0, a_1, \ldots) \mid a_i \in \mathbb{R}\}$. Show that this is a ring under the same operations as those defined for the polynomial ring $\mathbb{R}[x]$. This ring is called the ring of formal power series in x.

(3) Let $f(x) = x^3 + 3x - 2$ in $\mathbb{Q}[x]$.
 (a) (Not to be turned in) Show that $\mathbb{Q}[x]/(f(x))$ is a field.
 (b) Let u be the image of x in $\mathbb{Q}[x]/(f(x))$. Express the element $(u^2 - u + 4)^{-1}$ as a polynomial of degree at most two in u.

(4) (Ideals in a product ring)
 (a) Show that if I is an ideal of the product ring $R = \prod_{i=1}^{n} R_i$, then there are ideals $I_i \subset R_i$ for each i such that $I = \prod_{i=1}^{n} I_i$.
 (b) Use the Chinese remainder theorem to find all ideals of $\mathbb{Z}/60$.
 (c) Show by example that the analog of the first part is not true for groups. That is, find groups G_1 and G_2 and a normal subgroup $H \lhd G_1 \times G_2$ such that $H \ncong H_1 \times H_2$ for any normal subgroups $H_i \lhd G_i$.

(5) An element $r \in R$ (for any ring R) is called *nilpotent* if $r^n = 0$ for some $n \in \mathbb{N}$.
 (a) Show that if R is commutative, then the set of all nilpotent elements forms an ideal.
 (b) (Not to be turned in) Show that if R is any ring and $r \in R$ is nilpotent, then $1 + r$ is invertible.
 (c) Show that if R is a commutative ring without nilpotents, and $f(x) \in R[x]$ is a zerodivisor, then there is some nonzero $a \in R$ such that $af(x) = 0$.

(6) Consider the set $\mathscr{C}$ of infinite *Cauchy sequences* of rational numbers. This means that a sequence $(q_0, q_1, \ldots)$ is in $\mathscr{C}$ if and only if for every rational $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for every $m, n > N$, we have $|q_m - q_n| < \epsilon$.
 (a) Show that $\mathscr{C}$ is a unital commutative ring under the operations of component-wise addition and multiplication.
 (b) Let $\mathscr{Z} \subset \mathscr{C}$ be the set consisting of all sequences that converge to $0 \in \mathbb{Q}$. Recall that this means that a sequence $(q_0, q_1, \ldots)$ is in $\mathscr{Z}$ if and only if for every rational $\epsilon > 0$ there is some $N \in \mathbb{N}$ such thatn $|q_n| < \epsilon$ for each $n > N$. Show that $\mathscr{Z}$ is an ideal in $\mathscr{C}$.

(c) Show that $\mathscr{C}/\mathscr{Z}$ is a field, and that there is an injective map $\iota : \mathbb{Q} \to \mathscr{C}/\mathscr{Z}$.

(d) (Not to be turned in) Show that one can define an order on $\mathscr{C}/\mathscr{Z}$ by setting $(q_i) \geq (r_i)$ if and only if either $(q_i - r_i) \in \mathscr{Z}$, or there is some $N \in \mathbb{N}$ such that $q_n \geq r_n$ for each $n \in \mathbb{N}$. Show that this is a total order on $\mathscr{C}/\mathscr{Z}$ that satisfies the least upper bound property. We can thus show that $\mathscr{C}/\mathscr{Z} \cong \mathbb{R}$.