# MATH 8000 HOMEWORK 8

Due on Thursday, October 26

(1) Let $D$ be the set of complex numbers of the form $m + n\sqrt{-3}$ where $m$ and $n$ are either both integers or both half-integers (halves of odd integers).
   (a) Check that $D$ is a ring.
   (b) Show that $D$ is a Euclidean domain with respect to the Euclidean function $\delta(m+n\sqrt{-3}) = m^2 + 3n^2$.

(2) Prove that if $f(x)$ is a monic polynomial with integer coefficients, then any rational root of $f(x)$ is an integer.

(3) Let $F$ be a finite field with $q$ elements. Prove that the number of irreducible monic quadratic polynomials over $F$ equals $q(q-1)/2$, and the number of irreducible monic cubics is $q(q^2-1)/3$.

(4) Let $F$ be a subfield of $E$ and let $u \in E$ be algebraic over $E$ of odd degree over $F$. Show that $F(u) = F(u^2)$.

(5) Let $E/F$ be an algebraic (not necessarily finite) field extension: this means that every $x \in E$ is algebraic over $F$. Show that any subring of $E$ containing $F$ is a field.

(6) Let $E = F(u)$ where $u$ is transcendental over $F$. Let $K \neq F$ be a subfield of $E$ containing $F$. Show that $E$ is algebraic over $K$.

(7) Let $F$ be a subfield of $E$ and let $u, v \in E$ be algebraic over $F$. Suppose that the degrees of the minimum polynomials of $u$ and $v$ oven $F$ are relatively prime. Show that the minimum polynomial of $v$ is irreducible over $F(u)$.

(8) Show that if $E$ is a splitting field of $f(x)$ over $F$, and if $f(x)$ has degree $n$, then $[E : F] \leq n!$.