MATH 8000 HOMEWORK 9

Due on Thursday, November 2

(1) Let F be a field of characteristic not equal to 2.

(a) Let E be a quadratic extension of F, meaning that $[E : F] = 2$. Show that
$$S(E) = \{a \in F^\times \mid a \text{ is a square in } E\}$$
is a subgroup of $F^\times$ containing $(F^\times)^2 = \{a^2 \mid a \in F^\times\}$.

(b) Let E and E' be quadratic extensions of F. Show that there is an isomorphism $\varphi : E \to E'$ fixing F pointwise if and only if $S(E) = S(E')$.

(c) Show that there is an infinite sequence of fields $E_1, E_2, \dots$ with E_i a quadratic extension of $\mathbb{Q}$ such that E_i is not isomorphic to E_j for $i \neq j$.

(d) Let p be an odd prime. Show that up to isomorphism, there is exactly one extension of $\mathbb{F}_p$ that has p^2 elements.

(2) Find a splitting field of $X^{p^m} - 1$ over $\mathbb{F}_p$ for every $m \in \mathbb{N}$. What is its degree over $\mathbb{F}_p$?

(3) Let R be a commutative UFD. Prove *Eisenstein's irreducibility criterion*, stated as follows. Let
$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
be a polynomial in $R[x]$, and let $p \in R$ be a prime such that $p \mid a_i$ for each i but $p^2 \nmid a_0$. Then $f(x)$ is irreducible in $R[x]$. (Gauss' lemma implies that $f(x)$ is also irreducible over the fraction field of R.)

(4) Is the polynomial $x^3 + 4$ reducible or irreducible over $\mathbb{Q}[x]$? What about the polynomial $x^4 + 4$? Find the splitting fields of these polynomials over $\mathbb{Q}$, as subfields of $\mathbb{C}$.

(5) Prove that over any field, a polynomial $f(x)$ has multiple roots if and only $\gcd(f, f')$ is a non-constant polynomial. You may use the product rule for formal derivatives without proof.

(6) Let $f \in F[x]$, where F is a field of characteristic 0. Let $d(x) = \gcd(f, f')$. Show that $g(x) = f(x)d(x)^{-1}$ has the same roots as $f(x)$, and that these are all simple (multiplicity one) roots of $g(x)$.