THE LOWER CENTRAL SERIES QUOTIENTS OF A FREE ASSOCIATIVE SUPERALGEBRA

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1. Introduction

Let $A_n = \mathbb{C}\langle x_1, \ldots, x_n\rangle$ be the free associative algebra on n generators. Then A_n is also a Lie algebra with the commutator $[a, b] = ab - ba$. We can consider the *lower central series filtration* $L_0 \supseteq L_1 \supseteq L_2 \cdots$ of A_n, defined by $L_1(A_n) = A_n$ and $L_{i+1}(A_n) = [L_i(A_n), A_n]$. We also consider the associated graded algebra $B(A_n) = \bigoplus_i B_i$, where $B_i = L_i/L_{i+1}$.

The vector spaces B_i were first studied in [FS07] and later in [DE08] and [AJ09]. In this project, we generalise the results from these papers to the analogous results on superalgebras. Namely, let $A_{m,n}$ be the free superalgebra on m even and n odd generators. We can regard it as a Lie superalgebra with the commutator denoted by $\{a, b\} = ab - \epsilon_{a,b}ba$. Using this commutator, we can construct the lower central series filtration and its associated graded algebra as before. The eventual goal of this project is to generalise the results for A_n suitably so that they will be true for any symmetric tensor category (see, e.g. [Mac71]).

In [FS07], the authors prove that the vector space $B_2(A_n)$ is isomorphic to the even exact differential forms on $\mathbb{C}^n$, denoted by $\Omega^{ev}_{ex}(\mathbb{C}^n)$, and that a certain quotient $\overline{B_1}(A_n)$ of $B_1(A_n)$ is isomorphic to $\Omega^{ev}(\mathbb{C}^n)/\Omega^{ev}_{ex}(\mathbb{C}^n)$. In this paper, we outline the proof of the generalised result, which states the following:

Theorem 1.1. *Let $K_{m,n}$ be the space $A_{m,n}\{A_{m,n}, \{A_{m,n}, A_{m,n}\}\}A_{m,n}$. Then $A_{m,n}/K_{m,n}$ is isomorphic to $\Omega^{ev}(\mathbb{C}^{m|n})$ and $B_2(A_{m,n})$ is isomorphic to $\Omega^{ev}_{ex}(\mathbb{C}^{m|n})$. The space $\overline{B_1}(A_{m,n})$, defined as the quotient of $B_1(A_{m,n})$ by the image of $K_{m,n}$ in $B_1(A_{m,n})$, is isomorphic to $\Omega^{ev}(\mathbb{C}^{m|n})/\Omega^{ev}_{ex}(\mathbb{C}^{m|n})$.*

We then use this result to compute an explicit formula for the Hilbert series of $B_2(A_{m,n})$, as follows.

Theorem 1.2. *The Hilbert series of B_2 is given by the following formula:*

$$h(B_2) = \frac{1}{4} \cdot \prod_{i=1}^{m} \frac{(1+u_i)}{(1-u_i)} \cdot \prod_{j=1}^{n} \frac{(1+v_j)}{(1-v_j)} - \sum_{i=1}^{m} \frac{u_i}{2(1-u_i)} - \sum_{j=1}^{n} \frac{v_j}{2(1+v_j)} - \frac{1}{4}.$$

We then revisit the completely even setting of this problem and consider the lower central series quotients of A_n. It has been shown in [FS07] and [DE08] that each space $B_k(A_n)$ is a W_n-module which can be written as a direct sum of irreducible modules $\mathcal{F}_\lambda$, where λ is a Young diagram. In [AJ09], the authors prove a bound on the size $|\lambda|$ of such a Young diagram. We use this bound to prove a bound on $|\overline{\lambda}|$, which is the number of squares of λ that do not lie on the first column. Our bound

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is uniform for all $B_k(A_n)$ with a fixed k. Our result implies that as n grows, the corresponding Young diagrams for $B_k(A_n)$ only grow in the first column, because the number of squares away from the first column is finite and independent of n.

We have used the computational algebra system *MAGMA* to compute the dimensions of the multi-graded components of $B_k(A_{m,n})$ for some small values of k, m, n. From these computations we have conjectured a rational form for the Hilbert series for some of them. We have also formulated the following conjecture:

Conjecture 1.3. *The Hilbert series of $B_k(A_{m,n})$ is a rational function for any k, m, n.*

Additionally, we have formulated some more general conjectures about the structure of $B_k(A_{m,n})$. Let $W_{m,n}$-mod be the category of $W_{m,n}$ modules and let $\mathcal{K}$ be the Grothendieck group of $W_{m,n}$-mod. We define $[\mathcal{G}_\lambda]$ to be certain elements of $\mathcal{K}$, and conjecture the following:

Conjecture 1.4. *For all $B_k(A_{m,n})$, we can write the following decomposition in $\mathcal{K}$:*

$$[B_k(A_{m,n})] = \sum_{\lambda \in S} [\mathcal{G}_\lambda],$$

where S is a finite set consisting of Young diagrams λ that fit in the (m, n)-hook.

In addition, we have formulated the following conjecture based on computations that we carried out in MAGMA:

Conjecture 1.5. *The conjectured decompositions of $[B_k(A_{1,1})]$ for $k = 3, 4, 5$ are as follows.*

$$\begin{aligned}
[B_3(A_{1,1})] &= [\mathcal{G}_{(2,1)}],\\
[B_4(A_{1,1})] &= [\mathcal{G}_{(3,1)}] + [\mathcal{G}_{(2,1,1)}],\\
[B_5(A_{1,1})] &= [\mathcal{G}_{(4,1,1)}] + [\mathcal{G}_{(4,1)}] + [\mathcal{G}_{(3,1,1,1)}] + [\mathcal{G}_{(3,1,1)}] + [\mathcal{G}_{(2,1,1,1)}].
\end{aligned}$$

In Section 2 we provide some background material. In Section 3 we sketch the proof of Theorem 1.1. In Section 4 we use the result from Section 3 to explicitly compute the Hilbert series of $B_2(A_{m,n})$. In Section 5 we prove an upper bound on the size of $\overline{\lambda}$ in a Young diagram λ for which $\mathcal{F}_\lambda$ occurs in the Jordan-Hölder series of $B_k(A_n)$. In Section 6 we describe our computations and the conjectures that we formulated.

2. Background

In this section, we briefly recall some terms and notation that we will need. We begin by defining supervector spaces and superalgebras.

Definition 2.1. *A* supervector space *V is a vector space with a $(\mathbb{Z}/2\mathbb{Z})$-grading; $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$ are called the even and the odd parts of V respectively. A morphism of supervector spaces is a vector space morphism that preserves the degree in the $(\mathbb{Z}/2\mathbb{Z})$-grading.*

Definition 2.2. *A* superalgebra *A is both an algebra and a supervector space, such that $A_i \cdot A_j \subseteq A_{i+j}$ for every $i, j \in \mathbb{Z}/2\mathbb{Z}$.*

An element a in a graded algebra is called *homogeneous* if it is entirely contained in one of the graded pieces of A. When A is a superalgebra and $a \in A$ is homogeneous, we let $|a|$ denote the degree of a (either $\bar{0}$ or $\bar{1}$). If a, b are homogeneous elements of a superalgebra, we let $\epsilon_{a,b} = (-1)^{|a||b|}$.

If V and W are supervector spaces, the tensor product $V \otimes W$ is also a supervector space, with the following decomposition:

$$(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}),$$
$$(V \otimes W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}}).$$

Let V be a supervector space. The tensor superalgebra $T(V)$ on V is defined in the usual way:

$$T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}.$$

The superalgebra structure on $T(V)$ is the one induced from each summand $V^{\otimes i}$. $T(V)$ is also $\mathbb{Z}$-graded; the homogeneous elements of degree n are the tensors of rank n. For clarity, we will refer to the degree corresponding to the $\mathbb{Z}$-grading as the $\mathbb{Z}$-degree.

The symmetric algebra on V, denoted by $S(V)$, is the quotient of $T(V)$ by the two-sided ideal generated by

$$\langle a \otimes b - \epsilon_{a,b} b \otimes a \mid a \in V_i, b \in V_j \rangle.$$

The alternating, or exterior, algebra on V, denoted by $\Lambda(V)$, is the quotient of $T(V)$ by the two-sided ideal generated by

$$\langle a \otimes b + \epsilon_{a,b} b \otimes a \mid a \in V_i, b \in V_j \rangle.$$

We denote by $\mathbb{C}^{m|n} = \mathbb{C}^m_{\bar{0}} \oplus \mathbb{C}^n_{\bar{1}}$ the supervector space whose even part is $\mathbb{C}^m$ and whose odd part is $\mathbb{C}^n$.

Example 2.3. *The following example illustrates the definitions of $S(V)$ and $\Lambda(V)$ for a supervector space V.*

1. *Let $V = \mathbb{C}^{m|0}$. The space $S(V)$ is the polynomial algebra over $\mathbb{C}$ with m generators. The space $\Lambda(V)$ is the alternating algebra over $\mathbb{C}$ with m generators.*
2. *Let $V = \mathbb{C}^{0|n}$. The space $S(V)$ is the alternating algebra over $\mathbb{C}$ with n generators. The space $\Lambda(V)$ is the polynomial algebra over $\mathbb{C}$ with n generators.*

We let $A_{m,n}$ denote $T(\mathbb{C}^{m|n})$. That is, A is the free superalgebra with m even generators n odd generators.

Remark 2.4. The algebra $A_{m,n}$ is the same algebra as A_{m+n}, but we regard it as a superalgebra.

Definition 2.5. *A* Lie superalgebra $\mathfrak{A}$ *is a supervector space with a morphism $\{\cdot,\cdot\}\colon \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$ of supervector spaces such that:*

- *Super skew-symmetry: $\{x, y\} = -\epsilon_{x,y}\{y, x\}$ for every homogeneous x, y in $\mathfrak{A}$.*
- *The super Jacobi identity:*

$$\epsilon_{z,x}\{x, \{y, z\}\} + \epsilon_{x,y}\{y, \{z, x\}\} + \epsilon_{y,z}\{z, \{x, y\}\} = 0,$$

for every homogeneous x, y, z in $\mathfrak{A}$.

It is clear that an associative superalgebra becomes a Lie superalgebra with bracket $\{x, y\} = x \cdot y - \epsilon_{x,y} y \cdot x$.

Definition 2.6. *The* lower central series *of a Lie superalgebra $\mathfrak{A}$ is defined as the filtration*

$$\mathfrak{A} = L_1(\mathfrak{A}) \supseteq L_2(\mathfrak{A}) \supseteq L_3(\mathfrak{A}) \supseteq \cdots,$$

where $L_{i+1}(\mathfrak{A}) = \{L_i(\mathfrak{A}), \mathfrak{A}\}$.

Notice that $\{L_i, L_j\} \subset L_{i+j}$ for each i, j.

Definition 2.7. *Given a Lie superalgebra $\mathfrak{A}$, the* associated graded *Lie superalgebra $B(\mathfrak{A})$ is defined as*

$$B(\mathfrak{A}) = \bigoplus_{i=1}^{\infty} B_i(\mathfrak{A}),$$

where $B_i(\mathfrak{A}) = L_i(\mathfrak{A})/L_{i+1}(\mathfrak{A})$.

The spaces $L_i(A_{m,n})$ and $B_i(A_{m,n})$ will be the main focus of the remainder of this paper. For short, we will denote these by L_i and B_i respectively.

3. A Generalisation of the Feigin-Shoikhet Isomorphism

Feigin and Shoikhet proved the following facts in [FS07].

Theorem 3.1 ([FS07])**.** *Let K_n be the space $A_n[A_n, [A_n, A_n]]A_n$. Then A_n/K_n is isomorphic to $\Omega^{ev}(\mathbb{C}^n)$, the algebra of differential forms on $\mathbb{C}^n$. The space $B_2(A_n)$ is isomorphic to the exact forms $\Omega^{ev}_{ex}(\mathbb{C}^n)$. The quotient $\overline{B_1}(A_n)$ of $B_1(A_n)$ by the image of K_n in $B_1(A_n)$ is isomorphic to $\Omega^{ev}(\mathbb{C}^n)/\Omega^{ev}_{ex}(\mathbb{C}^n)$.*

This section generalises these results to the algebra $A_{m,n}$.

We first define the analogous notion of differential forms on a general supervector space.

Definition 3.2. *Let V be a supervector space. The algebra $\Omega(V)$ of* differential forms on V *is defined as*

$$\Omega(V) = S(V) \otimes \Lambda(V).$$

Remark 3.3. If A and B are superalgebras, the multiplication in the tensor product $A \otimes B$ is defined on homogeneous elements $a, c \in A$ and $b, d \in B$ as

$$(a \otimes b) \cdot (c \otimes d) = \epsilon_{b,c}(ac \otimes bd).$$

We will sometimes denote the product in $\Lambda(V)$ by the $\wedge$ sign and call it the *wedge* product. Also, we will usually drop the $\otimes$ sign and write $a \otimes b$ as $a \cdot b$ or ab.

The space $\Omega(V)$ is also $\mathbb{Z}$-graded by the degree of the form, which is the degree of the portion that comes from $\Lambda(V)$. For example, the subspace $S(V)$ of $\Omega(V)$ lies in degree 0, and is called the space of 0-forms. We will denote the degree corresponding to this grading of an element a by $\deg(a)$.

The space $\Omega(V)$ is equipped with a linear map $d \colon \Omega(V) \to \Omega(V)$ defined as follows. On the space $S^1(V)$, the map d sends every element to the corresponding element in $\Lambda^1(V)$, via the identification $S^1(V) = \Lambda^1(V) = V$. On $\Lambda^1(V)$, the map d sends every element to 0.

Remark 3.4. $\Omega(V)$ is generated as an algebra by $S^1(V) = V$ and $d(V)$.

Using the preceding remark, the map d is extended to $\Omega(V)$ by the following rule: $d(a \cdot b) = d(a) \cdot b + (-1)^{\deg(a)} a \cdot d(b)$. Notice that $\deg(d(a)) = \deg(a) + 1$, where for convenience we regard 0 to have degree ∞.

Lemma 3.5. *The composition $d \circ d\colon \Omega(V) \to \Omega(V)$ is the zero map.*

Proof. We know that d is zero on $\Lambda^1(V)$, so $d^2 = d \circ d$ is zero on $\Lambda^1(V)$. Also, we know that $d(S^1(V)) = \Lambda^1(V)$ so $d^2(S^1(V)) = 0$.

We now prove the result by induction on the degree of the form. Suppose that d^2 is zero on all forms of degree at most k. Let a be a 1-form and let b be a k-form. Then we have:

$$\begin{aligned} d^2(ab) &= d(d(a) \cdot b + (-1)^{\deg(a)} a \cdot d(b)), \\ &= d^2(a) \cdot b + (-1)^{\deg(da)} d(a) \cdot d(b) + (-1)^{\deg(a)} d(a) \cdot d(b) + a \cdot d^2(b), \\ &= 0, \end{aligned}$$

because $d^2(a) = d^2(b) = 0$ and $\deg(d(a)) = \deg(a) + 1$. Therefore the result is true for all forms of degree $k+1$. By induction, d^2 is the zero map. □

A element ω of $\Omega(V)$ is called a *closed* form if $d(\omega) = 0$. An element ω of $\Omega(V)$ is called an *exact* form if there exists an element a of $\Omega(V)$ such that $d(a) = \omega$. For forms of degree greater than 0, these two notions are equivalent by the well-known Poincaré lemma for the super de Rham complex.

The space $S(V) \otimes \Lambda^{ev}(V)$ of even forms on V is denoted by $\Omega^{ev}(V)$.

Remark 3.6. The space $\Omega^{ev}(V)$ is a super-commutative algebra. Namely,

$$a \cdot b = \epsilon_{a,b} b \cdot a,$$

for a, b that are homogeneous by the $(\mathbb{Z}/2\mathbb{Z})$-grading.

We define the *Fedosov product* on $\Omega(V)$ as follows. For any two elements a, b of $\Omega(V)$,

$$a * b = a \cdot b + (-1)^{\deg(a)} da \cdot db.$$

It is easy to check that this product is associative. The space $\Omega(V)$ with the Fedosov product is denoted by $\Omega_*(V)$.

Now we restate the generalisation of the Feigin-Shoikhet result to $A_{m,n}$.

Theorem 1.1. *Let $K_{m,n}$ be the space $A_{m,n}\{A_{m,n}, \{A_{m,n}, A_{m,n}\}\}A_{m,n}$. Then $A_{m,n}/K_{m,n}$ is isomorphic to $\Omega^{ev}(\mathbb{C}^{m|n})$ and $B_2(A_{m,n})$ is isomorphic to $\Omega^{ev}_{ex}(\mathbb{C}^{m|n})$. The space $\overline{B_1}(A_{m,n})$, defined as the quotient of $B_1(A_{m,n})$ by the image of $K_{m,n}$ in $B_1(A_{m,n})$, is isomorphic to $\Omega^{ev}(\mathbb{C}^{m|n})/\Omega^{ev}_{ex}(\mathbb{C}^{m|n})$.*

Let $A = A_{m,n}$ and $\Omega^{ev} = \Omega^{ev}(\mathbb{C}^{m|n})$. We denote the generators of $\mathbb{C}^{m|n}_{\bar{0}}$ by $z_1, \ldots, z_{m+n}$ and the generators of $\mathbb{C}^{m|n}_{\bar{1}}$ by $z_{m+1}, \ldots, z_{m+n}$.

Definition 3.7 (Feigin-Shoikhet map)**.** *We define the* Feigin-Shoikhet map φ *as follows:*

$$\begin{aligned} \varphi\colon A &\to \Omega^{ev}_*, \\ z_i &\mapsto z_i. \end{aligned}$$

We can now prove the following lemma.

Lemma 3.8. *The map φ is surjective.*

Proof. We first prove that all 0-forms in Ω^{ev}_* occur in the image of A, as follows. We know by definition that the generators $\{z_i\}$ occur. Now suppose that the monomials m_1 and m_2 occur as the images of a_1 and a_2 respectively. Then the monomial $m_1 \cdot m_2$ occurs as the image of $\frac{1}{2}(a_1a_2 + \epsilon_{a_1,a_2}a_2a_1)$. By induction, all monomials occur in the image of A and by linearity, all 0-forms occur in the image of A.

For all i, j, the form $dz_i \wedge dz_j$ occurs in Ω^{ev}_* as the image of $\frac{1}{2}\{z_i, z_j\}$. Therefore the wedge product of any two generators occurs in the image of A.

We know that a general form ω in Ω^{ev}_* is a linear combination of terms of the form $f \cdot dz_1 \cdots dz_{2k}$, where f is some 0-form and $z_1, \ldots, z_{2k}$ are distinct generators of A. But we can write such a term using the Fedosov product as

$$f * (dz_1 \wedge dz_2) * \cdots * (dz_{2k-1} \wedge dz_{2k}).$$

Since each piece of the preceding Fedosov product occurs in the image of A, the form ω must also occur in the image of A. Thus the proof is complete. $\square$

Outline of proof of Theorem 1.1. We first show that φ induces the isomorphism $A/K \cong \Omega^{ev}_*$. A straightforward computation shows that $\varphi(\{A, \{A, A\}\}) = 0$, so K belongs to the kernel of φ.

To show that K is exactly the kernel of φ, we will show that the induced map $\varphi' \colon A/K \to \Omega^{ev}_*$ is an isomorphism. We now consider a presentation of Ω^{ev}_* by generators and relations, and show that φ' is an invertible map that preserves the relations.

Let $\eta_{i,j}$ denote the form $2dz_i \wedge dz_j$ in Ω^{ev}_*. Let $X = \{z_1, \ldots, z_{m+n}\}$. The space of all even forms Ω^{ev} is generated by X, subject to the following relations:

(1) $a \cdot b = \epsilon_{a,b} b \cdot a$ for all $a, b \in X$.
(2) $\eta_{i,j} = -\epsilon_{z_i,z_j}\eta_{j,i}$ for all i, j.
(3) $\eta_{i,j} \cdot \eta_{k,l} = -\epsilon_{z_j,z_k}\eta_{i,k} \cdot \eta_{j,l}$ for all i, j, k, l.

We know that φ maps z_i to z_i, and $\{z_i, z_j\}$ to $\eta_{i,j}$. We can check the preceding relations as follows.

(1) The super-commutativity checks are straightforward.
(2) For all z_i, z_j in A, the following is true by definition:

$$\{z_i, z_j\} + \epsilon_{z_i,z_j}\{z_j, z_i\} = 0.$$

If we apply φ' to the identity above, we get $\eta_{i,j} = -\epsilon_{z_i,z_j}\eta_{j,i}$.
(3) For all z_i, z_j, z_k, z_l in A, we have the following identity:

$$\begin{aligned}\{z_i, z_j\} \cdot \{z_k, z_l\} + \epsilon_{z_j,z_k}\{z_i, z_k\} \cdot \{z_j, z_l\} \\ = \epsilon_{z_i,z_j}\epsilon_{z_i,z_k}\{\{z_j, z_k\}, z_iz_l\} + \epsilon_{z_j,z_k}z_i\{z_k, \{z_j, z_l\}\} \\ + \{\{z_i, z_j\}, z_k\}z_l - \epsilon_{z_j,z_k}\epsilon_{z_j,z_l}\epsilon_{z_k,z_l}\{\{z_iz_l, z_k\}, z_j\}.\end{aligned}$$

Notice that each of the terms on the right hand side of the identity belongs to K, so the left hand side also belongs to K. If we apply φ' to this identity, we get $\eta_{i,j} \cdot \eta_{k,l} = -\epsilon_{z_j,z_k}\eta_{i,k} \cdot \eta_{j,l}$.

Remark 3.9. Each identity above is a super analogue of the identities in [FS07].

Therefore $\operatorname{Ker}(\varphi') = 0$, and so $A/K \cong \Omega^{ev}_*$.

We now show that the spaces $\overline{B_1}(A)$ and $\Omega^{ev}/\Omega^{ev}_{ex}$ are isomorphic. Notice that $\varphi(\{a, b\}) = 2da \wedge db$, which is exact because $d(a \cdot db) = da \wedge db$. Therefore the image of L_2 under φ lies in Ω^{ev}_{ex}. By a proof that is similar to the one above, we can show that $\varphi\,|_{L_2}$ is surjective onto Ω^{ev}_{ex}.

We know that $B_1(A) = A/L_2$ and also that $A/K \cong \Omega^{ev}$. From this it is clear that $\overline{B_1}(A) \cong \Omega^{ev}/\Omega^{ev}_{ex}$.

To prove that $B_2(A) \cong \Omega^{ev}_{ex}$, it is enough to prove the following lemma.

Lemma 3.10. *The intersection $L_2 \cap K$ is L_3.*

Assuming Lemma 3.10, we can argue as follows. B_2 is the quotient L_2/L_3. But we know that $L_2/K = \Omega^{ev}_{ex}$. So

$$B_2 = L_2/L_3 = L_2/(L_2 \cap K) = L_2/K = \Omega^{ev}_{ex}.$$

Therefore proving that $B_2(A) \cong \Omega^{ev}_{ex}$ reduces to proving Lemma 3.10.

The proof of the analogue of Lemma 3.10 as given in [FS07] is based on the representation theory of $\mathfrak{gl}(n)$. In particular, it is based on an application of Schur-Weyl duality. We can apply the same arguments to $\mathfrak{gl}(m|n)$, but we omit them. □

4. The Hilbert series of $B_2(A_{m,n})$

We have outlined a proof of the fact that $B_2(A_{m,n})$ is isomorphic to the even exact forms $\Omega^{ev}_{ex}(\mathbb{C}^{m|n})$. For the remainder of the paper, we will assume this result. In this section, we will construct an explicit formula for the Hilbert series for B_2.

Definition 4.1. *Let $R = \bigoplus_{\vec{i}} R_{\vec{i}}$ be a $(\mathbb{Z}^{\geq 0})^m$-graded algebra such that each $R_{\vec{i}}$ is finite-dimensional. Then the* Hilbert series *of R, denoted by $h(R)$, is the following formal power series:*

$$h(R) = \sum_{\vec{i}} \dim(R_{\vec{i}}) t_1^{i_1} \cdots t_m^{i_m}.$$

We know that $B_k(A_{m,n})$ is $(\mathbb{Z}^{\geq 0})^{m+n}$-graded by the multi-degrees of the generators. Moreover, it is clear that each graded piece is finite-dimensional, because $B_k(A_{m,n})$ is a subquotient of the free superalgebra $A_{m,n}$ for every k. Therefore the Hilbert series of $B_k(A_{m,n})$ is well-defined.

In the Hilbert series of B_2, let the variables $u_1, \ldots, u_m$ correspond to the even generators and let $v_1, \ldots, v_n$ correspond to the odd generators.

Theorem 1.2. *The Hilbert series of B_2 is given by the following formula:*

$$h(B_2) = \frac{1}{4} \cdot \prod_{i=1}^{m} \frac{(1+u_i)}{(1-u_i)} \cdot \prod_{j=1}^{n} \frac{(1+v_j)}{(1-v_j)} - \sum_{i=1}^{m} \frac{u_i}{2(1-u_i)} - \sum_{j=1}^{n} \frac{v_j}{2(1+v_j)} - \frac{1}{4}.$$

Proof. We begin by noting that

$$h(B_2) = h(\Omega^{ev}_{ex}) = \sum_{i=0}^{\infty} h(\Omega^{2i}_{ex}).$$

The map $d\colon \Omega(\mathbb{C}^{m|n}) \to \Omega(\mathbb{C}^{m|n})$ gives rise to the following complex:

$$\cdots \xrightarrow{d} \Omega^{i-1} \xrightarrow{d} \Omega^{i} \xrightarrow{d} \Omega^{i+1} \xrightarrow{d} \cdots.$$

This sequence is exact except for a copy of $\mathbb{C}$ at $i = 0$. For all other i, we can write the following:

$$\Omega^{i+1}/d(\Omega^i) \cong d(\Omega^{i+i}), \text{ or equivalently,}$$
$$\Omega^{i+1}/\Omega^{i+1}_{ex} \cong \Omega^{i+2}_{ex}.$$

Therefore we have the following equation for all $i \geq 2$:

$$h(\Omega^i_{ex}) = h(\Omega^{i-1}) - h(\Omega^{i-1}_{ex}). \tag{1}$$

We can rewrite the above formula as

$$h(\Omega^i_{ex}) = \sum_{j=0}^{i-1} (-1)^{i-j+1} h(\Omega^j) + (-1)^{i+1}. \tag{2}$$

We know that $\Omega(\mathbb{C}^{m|n}) = S(\mathbb{C}^{m|n}) \otimes \Lambda(\mathbb{C}^{m|n})$. From this,

$$h(\Omega) = \frac{\prod_{j=1}^n (1+v_j)}{\prod_{i=1}^m (1-u_i)} \cdot \frac{\prod_{i=1}^m (1+tu_i)}{\prod_{j=1}^n (1-tv_j)},$$

where the variable t is a counter for the degree of the form. Therefore $h(\Omega^i)$ is just the coefficient of t^i in $h(\Omega)$. From Equation 1 and Equation 2, we can write the following:

$$\begin{aligned} h(\Omega^{2k}_{ex}) &= \sum_{i=0}^{2k-1} (-1)^{i+1} \mathrm{Coeff}_{t^i}(h(\Omega)) + 1, \\ &= \mathrm{Res}_{t=0} \left(h(\Omega) \cdot \frac{1}{t} \cdot \frac{t^{-2k} - 1}{1 + t^{-1}} \right) + 1. \end{aligned}$$

Notice that the term $(t^{-2k} - 1)/(1 + t^{-1})$ does not converge as a power series unless $|t| > 1$. So we can compute this residue by taking a contour integral around a circle γ with centre at the origin and radius greater than 1. Therefore we can write $h(\Omega^{2k}_{ex})$ as

$$h(\Omega^{2k}_{ex}) = \frac{1}{2\pi i} \int_\gamma \left(h(\Omega) \cdot \frac{-1}{1+t} \right) + \frac{1}{2\pi i} \int_\gamma \left(h(\Omega) \cdot \frac{t^{-2k}}{1+t} \right) + 1.$$

An elementary computation shows that

$$\frac{1}{2\pi i} \int_\gamma \left(h(\Omega) \cdot \frac{-1}{1+t} \right) = -1.$$

So we get

$$h(\Omega^{2k}_{ex}) = \frac{1}{2\pi i} \int_\gamma \left(h(\Omega) \cdot \frac{t^{-2k}}{1+t} \right).$$

Notice that there are no exact forms in degree 0. So we can consider $h(\Omega^{ev}_{ex})$ to be the sum $\sum_{k \geq 1} h(\Omega^{2k}_{ex})$, which simplifies to the following:

$$h(\Omega^{ev}_{ex}) = \frac{1}{2\pi i} \int_\gamma \left(h(\Omega) \cdot \frac{1}{(1+t)(1-t^{-2})} \right) - 1.$$

The integrand has a simple pole at $t = 1$ and a double pole at $t = -1$. We can calculate the integral by the method of residues (see, e.g. [Ahl79]). This gives us the required formula. $\square$

5. An Upper Bound for $\mathcal{F}_\lambda$

In this section, we focus on the completely even case, namely we consider $A = A_n = A_{n,0}$. We will build on some previous results to establish an upper bound on the number of squares away from the first column in all Young diagrams λ for which the module $\mathcal{F}_\lambda$ occurs in $B_k(A)$.

Remark 5.1. In any symmetric tensor category, the symmetric group S_n acts on n-fold tensor products in the natural way. Thus it makes sense to define Schur functors $\mathbb{S}_\lambda$ in symmetric tensor categories as multiplicity spaces of the irreducible S_n module W_λ, just as one does with $\mathfrak{gl}(m)$ in the construction of Schur-Weyl duality (see, e.g. [FH91]). In the coming months, we plan to develop the study of lower central series in the generality of symmetric tensor categories, which in particular encompass all of the categories $\mathfrak{gl}(m|n)$-mod considered in the present note.

It is more or less straightforward to see that the key constructions for lower central series (tensor algebras, associated graded components, differential forms, etc.) make perfect sense in any symmetric tensor category, and that each B_m can be built from Schur functors using the symmetric group action.

It is well-known that Schur functors $\mathbb{S}_\lambda$ are zero for $\mathfrak{gl}(m)$ if and only if λ contains more than m rows. Thus when m is sufficiently large, each Schur functor $\mathbb{S}_\lambda$ is faithful $\mathfrak{gl}(m)$-mod. This means that the bounds we prove here in the completely even setting will apply to arbitrary symmetric tensor categories, and in particular to all the categories $\mathfrak{gl}(m|n)$-mod.

Definition 5.2. *Let $W_n = Der(\mathbb{C}[x_1, \ldots, x_n])$ denote the Lie algebra of* polynomial vector fields*:*

$$W_n = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \ldots, x_n]\partial_i,$$

with the Lie bracket defined as

$$[p\partial_i, q\partial_j] = p\frac{\partial q}{\partial x_i}\partial_j - q\frac{\partial p}{\partial x_j}\partial_i.$$

We know from [FS07] that $B_k(A)$ is a W_n-module for each $k \geq 2$. By combining results from [FS07] and [DE08], we know that the composition factors occurring in the Jordan-Hölder series of $B_k(A)$ are the irreducible modules $\mathcal{F}_\lambda$ of W_n, each occurring with finite multiplicity.

The main theorem from [AJ09] establishes a bound on the size of λ for those $\mathcal{F}_\lambda$ that occur in the Jordan-Hölder series of $B_k(A)$. The theorem is the following.

Theorem 5.3 ([AJ09])**.** *Let $k \geq 3$.*

(1) *For $\mathcal{F}_\lambda$ in the Jordan-Hölder series of $B_k(A)$,*
$$|\lambda| \leq 4k - 7 + 2\left\lfloor \frac{n-2}{2} \right\rfloor.$$
(2) *Let $n = 2$ or 3. For $\mathcal{F}_\lambda$ in the Jordan-Hölder series of $B_k(A)$,*
$$|\lambda| \leq 2k - 3.$$

Given a Young diagram λ, let $\overline{\lambda}$ be the Young diagram obtained by deleting the first column of λ. The following theorem gives a bound on the size of $\overline{\lambda}$ that is independent of the number of variables n. The significance of this is that we get only a finite collection of possible diagrams $\overline{\lambda}$, which may only be extended in the first column. The independence on n, combined with Remark 5.1 implies that this same finite collection of diagrams and their extensions in the first column describe each $B_k(A_{m,n})$ as well.

Theorem 5.4. *Let $k \geq 3$.*

(1) *For $\mathcal{F}_\lambda$ in the Jordan-Hölder series of B_k, we have*
$$|\overline{\lambda}| \leq 4k - 9,$$
(2) *Let n be 2 or 3. For $\mathcal{F}_\lambda$ in the Jordan-Hölder series of B_k, we have*
$$|\overline{\lambda}| \leq 2k - 5.$$

Proof. Let $n = 2$ or 3. We know from Theorem 5.3 that if $\mathcal{F}_\lambda$ occurs in $B_k(A)$, then $|\lambda| \leq 2k - 3$. Moreover, the modules $\mathcal{F}_{(r)}$ do not appear in $B_k(A_2)$ for $k \geq 3$, because they correspond to the generator z_1^r, which does not occur in $B_k(A_2)$. Therefore λ must have at least two rows. From this it is clear that $|\overline{\lambda}| \leq 2k - 5$.

For $n \geq 4$, we prove the result by induction. Suppose that $p - 1 \geq 2$ and that all $\mathcal{F}_\lambda$ occurring in $B_k(A_{p-1})$ satisfy the bound. Now consider some $\mathcal{F}_\lambda$ which occurs in $B_k(A_p)$.

If λ has less than p rows, let $\mu = (\lambda_1, \dots, \lambda_{p-1})$. Then $\mathcal{F}_\mu$ occurs in $B_k(A_{p-1})$. So $|\overline{\mu}| \leq 4k - 9$ by the induction hypothesis. The diagram λ cannot have more than p rows, because it is the Young diagram of a $\mathfrak{gl}(p)$ module. So we may assume that λ has p rows.

From the bound in Theorem 5.3, we deduce the following:
$$\lambda_1 + \cdots + \lambda_p \leq 4k - 7 + 2\left\lfloor \frac{p-2}{2} \right\rfloor \leq 4k - 7 + p - 2.$$
We know that $|\overline{\lambda}|$ is $\lambda_1 + \cdots + \lambda_p - p$. Using the preceding inequality, we get the desired bound:
$$|\overline{\lambda}| \leq 4k - 9.$$
□

Remark 5.5. It is conjectured that the bound in Theorem 5.3 can be improved to $2k - 3 + 2\lfloor \frac{n-2}{2} \rfloor$ for all n. Assuming this conjecture, the bound in Theorem 5.4 can also be improved to $2k - 5$ for all n. Improving the bound in Theorem 5.3 is one of the goals for future work on this subject.

6. Computations and Conjectures

We carried out several computations using the computational algebra system *MAGMA*, from which we were able to conjecture Hilbert series for $B_k(A_{m,n})$ for some small values of k, m, n. Additionally, we have formulated some more general conjectures that are supported by the computational data.

6.1. **Conjectures from Computations.** We now present some of the computations that we carried out using MAGMA. We computed the bi-graded Hilbert series for $B_k(A_{1,1})$ up to total degree 11 and the bi-graded Hilbert series for $B_k(A_{0,2})$ up to total degree 11. The dimensions of the graded pieces seem to stabilise when the total degree is high enough, for each B_k up to total degree 6. Beyond degree 6, it is not clear from the computed data that the dimensions stop growing.

Remark 6.1. We know a formula for the Hilbert series for $B_2(A_{m,n})$ from Section 4. Our computational data was consistent with the formula from Section 4.

The following result was proved in [AJ09]:

Theorem 6.2 ([AJ09])**.** *For all n, $B_3(A_n)$ can be decomposed as follows:*

$$B_3(A_n) = \bigoplus_{i=1}^{\infty}(2, 1^{2i-1}, 0^{n-2i}),$$

where $(2, 1^{2i-1}, 0^{n-2i})$ denotes the irreducible W_n-module $\mathcal{F}_\lambda$ corresponding to the Young diagram $\lambda = (2, 1^{2i-1}, 0^{n-2i})$ when $i \leq \lfloor \frac{n}{2} \rfloor$ and 0 otherwise.

Remark 6.3. Using Theorem 6.2 and the ideas outlined in Remark 5.1, we can prove a similar result for any symmetric tensor category, and in particular we can find the complete structure of $B_3(A_{m,n})$.

Observing that the dimensions of the graded pieces stop growing after a certain degree, we conjecture rational Hilbert series for $B_k(A_{1,1})$ and $B_k(A_{2,0})$, for $k = 4, 5$.

Conjecture 6.4. *Let $D = (1-x)^{-1}(1-y)^{-1}$. The conjectured Hilbert series for $B_k(A_{1,1})$ for $k = 4, 5$ are as follows:*

$$h(B_4(A_{1,1})) = xy(x+y)^2 \cdot D,$$
$$h(B_5(A_{1,1})) = xy(x+y)(x^2+y^2+xy+x^2y+xy^2) \cdot D.$$

The conjectured Hilbert series for $B_k(A_{0,2})$ for $k = 4, 5$ are as follows:

$$h(B_4(A_{0,2})) = xy(x^2+y^2+xy+x^2y^2) \cdot D,$$
$$h(B_5(A_{0,2})) = xy(x+y)(x^2+y^2+x^2y+xy^2+x^2y^2) \cdot D.$$

Based on the computational data, we also formulate the following conjecture:

Conjecture 1.3. *The Hilbert series of $B_k(A_{m,n})$ is a rational function for any k, m, n.*

6.2. **Definitions and Other Conjectures.** We first define some terms that we will need.

Definition 6.5. *Let $W_{m,n}$ denote the Lie superalgebra $Der(S(\mathbb{C}^{m|n}))$;*

$$W_{m,n} = S(\mathbb{C}^{m|n}) \otimes (\mathbb{C}^{m|n})^*.$$

Notice that $(\mathbb{C}^{m|n})^*$ is $(\mathbb{Z}/2\mathbb{Z})$-graded by picking a basis and assigning $|v^*| = |v|$ for homogeneous v. Then $W_{m,n}$ is also $(\mathbb{Z}/2\mathbb{Z})$-graded in the usual way. The bracket is defined in the same way as the bracket of W_n, with a sign correction corresponding to the $(\mathbb{Z}/2\mathbb{Z})$-grading.

Let $V = \mathbb{C}^{m|n}$. Let $W^0_{m,n}$ denote the subalgebra $V \otimes V^*$ of $W_{m,n}$. Let $W^{00}_{m,n}$ denote the subalgebra $\bigoplus_{k\geq 2} S^k(V) \otimes V^*$ of $W_{m,n}$.

Recall that the Lie algebra $\mathfrak{gl}(n)$ is the Lie algebra $\mathrm{End}(\mathbb{C}^n)$, with $[a,b] = ab - ba$.

Definition 6.6. *As a vector space, the Lie superalgebra $\mathfrak{gl}(m|n)$ is*

$$\mathfrak{gl}(m|n) = \mathfrak{gl}(m+n).$$

Let $\mathbb{C}^{m|n} = V_{\bar{0}} \oplus V_{\bar{1}}$. *Then*

$$\mathfrak{gl}(m|n)_{\bar{0}} = \mathrm{Hom}(V_{\bar{0}}, V_{\bar{0}}) \oplus \mathrm{Hom}(V_{\bar{1}}, V_{\bar{1}}),$$
$$\mathfrak{gl}(m|n)_{\bar{1}} = \mathrm{Hom}(V_{\bar{0}}, V_{\bar{1}}) \oplus \mathrm{Hom}(V_{\bar{1}}, V_{\bar{0}}).$$

The Lie superbracket is defined on homogeneous a, b *as*

$$\{A, B\} = AB - \epsilon_{A,B} BA.$$

The algebra $\mathfrak{gl}(m|n)$ is generated by the matrices $\{E_{i,j}\}$, where $E_{i,j}$ has 1 in the $(i,j)^{\text{th}}$ place and 0 everywhere else.

Lemma 6.7. *The quotient* $W^0_{m,n}/W^{00}_{m,n}$ *is isomorphic to* $\mathfrak{gl}(m|n)$.

Proof. The space $W^0_{m,n}/W^{00}_{m,n}$ is spanned by the elements $\{z_i \otimes z_j^*\}$, where $\mathbb{C}^{m|n}$ is generated by the even generators $z_1, \ldots, z_m$ and the odd generators $z_{m+1}, \ldots, z_{m+n}$. We define a map from $W^0_{m,n}/W^{00}_{m,n}$ to $\mathfrak{gl}(m|n)$ as follows:

$$z_i \otimes z_j^* \mapsto E_{i,j}.$$

It is clear from Definition 6.5 and Definition 6.6 that this map is an isomorphism. □

Let λ be a Young diagram. We will say that λ *fits in the* (a,b)*-hook* for non-negative integers a,b if $\lambda_{b+1} \le a$. For each λ that fits in the (m,n)-hook, we can define a corresponding irreducible module of $\mathfrak{gl}(m|n)$, denoted by V_λ (see, e.g. [MJ03]).

We can define $W_{m,n}$-modules $\mathcal{F}_\lambda$ that are analogous to the W_n-modules $\mathcal{F}_\lambda$. We do not discuss the $W_{m,n}$-action on these $\mathcal{F}_\lambda$ here, except to say that Lemma 6.7 is used to define this action. As modules of $\mathfrak{gl}(m|n)$, the may be defined as follows.

Definition 6.8. *If* λ *is a Young diagram that fits in the* (m,n)*-hook, then we define* $\mathcal{F}_\lambda$ *as a* $\mathfrak{gl}(m|n)$*-module as*

$$\mathcal{F}_\lambda = S(\mathbb{C}^{m|n}) \otimes V_\lambda.$$

Let λ and μ be Young diagrams. Then let $\lambda \circ \mu$ denote the diagram obtained by appending μ after λ, whenever the resulting diagram is a valid Young diagram. For example, $(3,2) \circ (2,1)$ is $(3,2,2,1)$.

Let λ be a Young diagram that fits in the (m,n)-hook. Let $\mathcal{F}_\lambda$ be the $W_{m,n}$-module as defined in Definition 6.8. Let $\mathcal{K}$ denote the Grothendieck group of the category $W_{m,n}$-mod. Given a $W_{m,n}$-module M, we will denote its image in $\mathcal{K}$ by $[M]$.

Definition 6.9. *We define* $[\mathcal{G}_\lambda] \in \mathcal{K}$ *as follows:*

$$[\mathcal{G}_\lambda] = \sum_{k=0}^{\infty} [\mathcal{F}_{\lambda \circ (1^{2k})}]$$

Conjecture 1.4. *For all* $B_k(A_{m,n})$, *we can write the following decomposition in* $\mathcal{K}$*:*

$$[B_k(A_{m,n})] = \sum_{\lambda \in S} [\mathcal{G}_\lambda],$$

where S *is a finite set consisting of Young diagrams* λ *that fit in the* (m,n)*-hook.*

To verify Conjecture 1.4, we studied the available data for the bi-graded Hilbert series for $B_k(A_{1,1})$. We computed the Hilbert series for $\mathcal{F}_\lambda$ for some λ that fit in the $(1,1)$ hook using the formula for the character of V_λ from [MJ03]. Comparing these two computations, we conjectured the decompositions of $[B_k(A_{1,1})]$ into a sum of $[\mathcal{G}_\lambda]$ for some small values of k.

Conjecture 1.5. *The conjectured decompositions of* $[B_k(A_{1,1})]$ *for* $k = 3, 4, 5$ *are as follows.*

$$\begin{aligned}
&[B_3(A_{1,1})] = [\mathcal{G}_{(2,1)}],\\
&[B_4(A_{1,1})] = [\mathcal{G}_{(3,1)}] + [\mathcal{G}_{(2,1,1)}],\\
&[B_5(A_{1,1})] = [\mathcal{G}_{(4,1,1)}] + [\mathcal{G}_{(4,1)}] + [\mathcal{G}_{(3,1,1,1)}] + [\mathcal{G}_{(3,1,1)}] + [\mathcal{G}_{(2,1,1,1)}].
\end{aligned}$$

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