SPHERICAL OBJECTS AND STABILITY CONDITIONS ON 2-CALABI–YAU QUIVER CATEGORIES

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Abstract. We study actions of spherical twists on 2-Calabi–Yau categories with a Bridgeland stability condition. In these categories, we describe how to reduce the phase spread of a spherical object using stable spherical twists. In 2-Calabi–Yau quiver categories, we describe how to construct all spherical stable objects by applying simple spherical twists to the simple objects. As applications, we give new proofs of the following theorems for 2-Calabi–Yau categories associated to ADE quivers: (1) all spherical objects lie in the braid group orbit of a simple object, and (2) the space of Bridgeland stability conditions is connected.

1. Introduction

Suppose we have a category $\mathcal{C}$ with a rich class of spherical objects. To what extent can we simplify an object of $\mathcal{C}$ by applying spherical twists? Dually, to what extent can we build complicated objects of $\mathcal{C}$ by applying spherical twists to simple objects? In this paper, we explore these questions for 2-Calabi–Yau triangulated categories using a measure of complexity provided by a Bridgeland stability condition. These questions are categorical analogues of similar questions in symplectic/algebraic geometry studied in [1, 14, 20].

Spherical objects play an important role in Calabi–Yau (CY) categories. Recall that an object of an n-CY category is spherical if its graded endomorphism ring is isomorphic to the cohomology ring of the n-sphere—this is the simplest possible endomorphism ring, given the n-CY condition. Under some finiteness and rigidity conditions on the category, a spherical object gives an auto-equivalence of the category, called a spherical twist [19].

Spherical objects interact wonderfully with Bridgeland stability conditions. For example, the stable Harder–Narasimhan (HN) factors of any spherical object with respect to a Bridgeland stability condition are themselves spherical (see, e.g. [11, Corollary 2.3]). In some cases (for example, for derived categories of K3 surfaces), a stability condition is determined by its behaviour on spherical objects (see [11]).

Our first result (Theorem 3.5) simplifies an object of a 2-CY triangulated category $\mathcal{C}$ by applying spherical twists. We show that the positive twist in the lowest HN factor and the negative twist in the highest HN factor decrease the phase spread (see Theorem 3.5 for a precise statement). Iterating this procedure gives a sequence of objects with smaller and smaller spread. If the set of possible spreads is discrete, then this procedure must stop, and then we have reduced our object to a stable object.

Having proved a general phase spread reduction statement, we focus on 2-CY categories $\mathcal{C}_\Gamma$ associated to a quiver Γ. These are triangulated categories arising in many places in symplectic geometry, algebraic geometry, and representation theory. Roughly speaking, they are the categories generated by a Γ-configuration of spherical objects (see § 2.3 for precise definitions). We give an explicit classification of the stable spherical objects of $\mathcal{C}_\Gamma$.

We now describe the classification of stable spherical objects. The category $\mathcal{C}_\Gamma$ is generated by spherical objects P_v as v ranges over the vertices of Γ. The Grothendieck group $K(\mathcal{C}_\Gamma)$ with the hom pairing is isomorphic to the root lattice of Γ. The braid group B_Γ acts on $\mathcal{C}_\Gamma$ by spherical twists in P_v, lifting the action of the Coxeter group on the root lattice. The extension closure of P_v is the heart of a bounded t-structure on $\mathcal{C}_\Gamma$ called the standard heart.

Let τ be a generic stability condition on $\mathcal{C}_\Gamma$ whose heart is the standard heart. Given a real root $w \in K(\mathcal{C}_\Gamma)$, it turns out that there are infinitely many spherical objects with class w. We prove that exactly one of them (up to triangulated shift) is τ-stable, and give an explicit procedure to construct it. The procedure goes as follows. Let w be a real root. Write w as a sequence of simple reflections applied to a simple root v, say

$$w = s_{v_n} \cdots s_{v_1} v,$$

and assume that this writing is minimal. Let $\sigma_v \colon \mathcal{C}_\Gamma \to \mathcal{C}_\Gamma$ be the spherical twist in P_v. We show that the unique τ-stable stable object P_w of class w can be written as

$$P_w = \sigma_{v_n}^{\epsilon_n} \cdots \sigma_{v_1}^{\epsilon_1} P_v,$$

for a particular choice of $\epsilon_i \in \{+1, -1\}$, which we describe explicitly. This choice is governed by how the root sequence for the minimal expression for w interacts with τ.

In the last section (§ 5), we give two applications of the simplification procedure and of the classification of spherical stable objects mentioned above. The first application is a new proof of the following theorem.

Theorem 1.1. *Let Γ be a quiver of type A_n, D_n, or E_6, E_7, E_8. The spherical objects of $\mathcal{C}_\Gamma$ lie in the B_Γ orbit of the simple objects of the standard heart.*

In the main text, Theorem 1.1 is Corollary 5.2. A proof of this theorem in type A appears in [13, 14] and may also follow for all ADE types from the ideas of [2].

The second application is a new proof of the following theorem.

Theorem 1.2. *Let Γ be a quiver of type A_n, D_n, or E_6, E_7, E_8. Any stability condition $\tau \in \mathrm{Stab}(\mathcal{C}_\Gamma)$ is in the B_Γ orbit of a standard stability condition. Furthermore, $\mathrm{Stab}(\mathcal{C}_\Gamma)$ is connected.*

In the main text, Theorem 1.2 is Corollary 5.5. Theorem 1.2 has been proved in [13, 14] for type A, and in [2] for types A, D, and E.

We highlight that the phase spread reduction works in general, not necessarily just for quiver categories. The classification of spherical stable objects works in any quiver category, not just those of finite type. The finite type assumption guarantees that the simplification procedure terminates. It is likely that termination works more generally (we have not found any counter-examples). We hope to address this question in the future.

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2. Background

In this section, we recall the notions of Bridgeland stability conditions, spherical objects and twists, and the CY categories associated to quivers.

2.1. **Bridgeland stability conditions.** Let $\mathcal{C}$ be a triangulated category. A *(Bridgeland) stability condition* on $\mathcal{C}$ consists of

(1) a homomorphism $Z\colon K(\mathcal{C}) \to \mathbf{C}$,
(2) an additive subcategory $\mathcal{P}(\phi) \subset \mathcal{C}$ for every $\phi \in \mathbf{R}$,

satisfying the following axioms:

(1) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;
(2) if $A \in P(\phi)$ and $B \in P(\psi)$ with $\phi > \psi$, then $\operatorname{Hom}(A, B) = 0$;
(3) for each nonzero object $X \in \mathcal{C}$, there is a diagram

$$\begin{tikzcd}
0 = X_0 \arrow[rr] & & X_1 \arrow[rr] \arrow[dl] & & X_2 \arrow[dl] & \cdots & X_{n-1} \arrow[rr] & & X_n = X, \arrow[dl] \\
& A_1 \arrow[ul, dashed, "+1"] & & A_2 \arrow[ul, dashed, "+1"] & & \cdots & & A_n \arrow[ul, dashed, "+1"] &
\end{tikzcd}$$

where the triangles are distinguished and $A_i \in \mathcal{P}(\phi_i)$ with

$$\phi_1 > \cdots > \phi_n;$$

(4) for every $A \in P(\phi)$, there is a positive real number $m(A)$ such that

$$Z([A]) = m(A) \cdot e^{\pi i \phi}.$$

The homomorphism Z is called the *central charge* and the collection of subcategories $\mathcal{P}$ is called the *slicing*. The objects in $\mathcal{P}(\phi)$ are said to be *semistable of phase* ϕ. Property (3) is called the *Harder–Narasimhan (HN) property*. The diagram asserted by the HN property is unique up to isomorphisms (see [5, § 3]). It is called the *HN filtration* of X. The objects A_i are called the *HN pieces*.

For us, the top and the bottom HN pieces play a key role. Given an object X, denote by $\lfloor X \rfloor = A_n$ the HN piece of X that has the lowest phase, and let $\phi^-(X)$ to this phase ϕ_n. Likewise, set $\lceil X \rceil = A_1$ and $\phi^+(X) = \phi_1$. We call $\lceil X \rceil$ (resp. $\lfloor X \rfloor$) the *top* (resp. *bottom*) of X. The *spread* of X is the difference $\phi^+(X) - \phi^-(X)$. The spread of X is zero if and only if X is semi-stable.

There are several abelian categories associated to a stability condition. First of all, the additive categories $\mathcal{P}(\phi)$ turn out to be abelian [5, Lemma 5.2]. For any interval $I \subset \mathbf{R}$, let $\mathcal{P}(I)$ be the additive category of $\mathcal{C}$ that is the extension closure of $\mathcal{P}(\phi)$ for $\phi \in I$. This is a "quasi-abelian" category (see [5, § 4] for the definition). If I is a half open interval of length 1, namely of the form $[\phi, \phi + 1)$ or $(\phi, \phi + 1]$, then $\mathcal{P}(I)$ turns out to be abelian. Moreover, it is the heart of a bounded t-structure on $\mathcal{C}$, which we call the *I-heart* of the stability condition.

A stability condition is uniquely determined by any of its I-hearts and the central charge. More precisely, suppose we are given the heart $\mathcal{A} \subset \mathcal{C}$ of a bounded t-structure and a linear map $Z\colon K(\mathcal{A}) \to \mathbf{C}$. Suppose that Z has the Harder–Narasimhan property and maps non-zero objects of $\mathcal{A}$ to the (semi-closed) upper half plane

$$\mathbf{H} = \{r \exp(i\pi\phi) \mid r > 0 \text{ and } 0 \leq \phi < 1\}.$$

Then there is a unique stability condition on $\mathcal{C}$ whose $[0, 1)$ heart is $\mathcal{A}$ and whose central charge is Z (see [5, Proposition 5.3]). If $\mathcal{A}$ is of finite length with finitely many simple objects

$A_1, \dots, A_n$ (up to isomorphism), then $K(\mathcal{A})$ is the free abelian group on $[A_1], \dots, [A_n]$. Then the central charge is uniquely specified by the values $Z([A_1]), \dots, Z([A_n])$. The finite length hypothesis implies that the Harder–Narasimhan property always holds.

2.2. **Spherical objects and spherical twists.** Fix a field $\mathbf{k}$. Let $\mathcal{C}$ be a $\mathbf{k}$-linear triangulated category. Assume that $\mathcal{C}$ is of finite type. That is, for any pair of objects $X, Y \in \mathcal{C}$, the vector space

$$\bigoplus_{i \in \mathbf{Z}} \operatorname{Hom}^i(X, Y)$$

is finite dimensional (here $\operatorname{Hom}^i(X,Y)$ stands for $\operatorname{Hom}(X, Y[i])$).

We say that $\mathcal{C}$ is d-Calabi–Yau (d-CY) if for every pair of objects $X, Y \in \mathcal{C}$, we have functorial isomorphisms

$$\operatorname{Hom}^i(X, Y) \cong \operatorname{Hom}^{d-i}(Y, X)^\vee.$$

Here the superscript $\vee$ stands for the $\mathbf{k}$-linear dual. An object X of a d-CY category $\mathcal{C}$ is called *spherical* if its graded endomorphism algebra is isomorphic to the cohomology algebra of the d-sphere

$$\bigoplus_{i \in \mathbf{Z}} \operatorname{Hom}^i(X, X) \cong H^*(S^d, \mathbf{k}).$$

Explicitly, the only non-trivial endomorphisms $X \to X[i]$ are the identity $X \to X$ and a map $X \to X[d]$, unique up to scaling.

Let $\mathcal{C}$ be a $\mathbf{k}$-linear d-CY category of finite type and let $X \in \mathcal{C}$ be a spherical object. Assume that $\mathcal{C}$ admits a dg enhancement, and that we have fixed such a dg enhancement. Such an enhancement exists if $\mathcal{C}$ is algebraic in the sense of [16] or enhanced in the sense of [4]. Associated to X is an auto-equivalence of $\mathcal{C}$

$$\sigma_X \colon \mathcal{C} \to \mathcal{C}$$

called the *spherical twist* in X, defined as follows (see [19, § 2.2]). Set $\operatorname{Hom}^*(X, Y) = \bigoplus_i \operatorname{Hom}^i(X, Y)$. For $Y \in \mathcal{C}$, the twist $\sigma_X(Y)$ is defined to be the cone of the evaluation map

$$X \otimes \operatorname{Hom}^*(X, Y) \to Y.$$

The dg enhancement of $\mathcal{C}$ allows a functorial cone. By definition, we have an exact triangle

$$\operatorname{Hom}^*(X, Y) \otimes X \to Y \to \sigma_X(Y) \xrightarrow{+1} . \tag{1}$$

We have a similar exact triangle for the inverse twist:

$$\sigma_X^{-1}(Y) \to Y \to X \otimes \operatorname{Hom}^*(Y, X)^\vee \xrightarrow{+1} . \tag{2}$$

2.3. **2-Calabi–Yau categories associated to quivers.** We recall the construction of a natural 2-CY category associated to a quiver. Our description uses the zig-zag algebra from [9]. In the case of the A_n-graph, the category was described in [22].

2.3.1. *The zig-zag algebra.* Fix a field $\mathbf{k}$. Let Γ be a finite connected graph without loops or multiple edges. Associated to Γ is an algebra $A(\Gamma)$ called the *zig-zag algebra*, constructed as follows. Assume, for simplicity, that Γ has at least 2 vertices. Let Γ^{dbl} be the directed graph obtained by doubling Γ, that is, by replacing each edge $v - w$ in Γ by a pair of edges $v \to w$ and $w \to v$. A path is a sequence of vertices $v_1, \dots, v_n$ such that $v_i \to v_{i+1}$ is an edge for all $i = 1, \dots, n-1$. We use the notation $(v_1|\dots|v_n)$ to denote a path. Recall that the path algebra is the $\mathbf{k}$-algebra generated as a $\mathbf{k}$-vector space by all paths, and where multiplication is induced by concatenation of paths. The zig-zag algebra $A(\Gamma)$ is the quotient of the path algebra of Γ^{dbl} by the two sided ideal generated by the following elements:

(1) paths of length 3,
(2) paths $(a|b|c)$ of length 2 if $a \neq c$,
(3) elements $(a|b|a) - (a|c|a)$ if a has edges to b and c.

It is easy to check that $A(\Gamma)$ is finite dimensional as a $\mathbf{k}$-vector space.

We endow $A(\Gamma)$ with the grading given by path length. Then $A(\Gamma)$ has non-zero graded components of degree 0, 1, and 2. For a vertex $v \in \Gamma$ let $e_v = (v)$ be the the length 0 path at v. Set $P_v = A(\Gamma)e_v$. Then P_v is a graded (left) $A(\Gamma)$-module. Furthermore, we have a decomposition

$$A(\Gamma) = \bigoplus_{v \in \Gamma} P_v$$

of left $A(\Gamma)$ modules. In particular, P_v is projective.

Let $\ell_v \in A(\Gamma)$ be the loop at v. This is the element of $A(\Gamma)$ represented by the path $(v|w|v)$ for any w adjacent to v (the zig-zag relations imply that different choices of w give the same element). An easy computation shows that the graded Hom spaces between the P_v are as follows

$$\mathrm{Hom}^i_A(P_v, P_w) = \begin{cases} \mathbf{k}\langle \mathrm{id} \rangle & \text{if } i = 0 \text{ and } v = w, \\ \mathbf{k}\langle \ell_v \rangle & \text{if } i = 2 \text{ and } v = w, \\ \mathbf{k}\langle (v|w) \rangle & \text{if } i = 1 \text{ and } v - w \text{ is an edge of } \Gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

2.3.2. *The category $\mathcal{C}_\Gamma$.* Consider $A = A(\Gamma)$ as a differential graded algebra (dga) with the zero differential. Let $K(\text{dgmod-}A)$ be the category whose objects are differential graded left A-modules (dgms) that are finite dimensional as $\mathbf{k}$-vector spaces, and whose morphisms are homotopy classes of maps of dgms. Then $K(\text{dgmod-}A)$ is a triangulated category in the standard way (see [21, § 22.8]). The category we are interested in is its smallest (full and strict) triangulated subcategory containing the objects P_v for $v \in \Gamma$:

$$\mathcal{C}_\Gamma = \langle P_v \mid v \in \Gamma \rangle \subset K(\text{dgmod-}A).$$

Our description of $\mathcal{C}_\Gamma$ differs slightly from the description in [22]; we now reconcile the two. Let $\mathcal{D}A$ be the localisation of $K(\text{dgmod-}A)$ obtained by inverting the quasi-isomorphisms.

Proposition 2.1. *The natural map $\mathcal{C}_\Gamma \to \mathcal{D}A$ is fully faithful.*

Proof. Let X be an object of $\mathcal{C}_\Gamma$. Since the objects P_v are summands of A, it follows that the complex X has property (P) in the sense of [15, § 3]. As a result, for any object $Y \in \mathcal{C}_\Gamma$, we have an equality

$$\mathrm{Hom}_{K(\text{dgmod-}A)}(X, Y) = \mathrm{Hom}_{\mathcal{D}A}(X, Y).$$

In particular, the map

$$\mathcal{C}_\Gamma \to \mathcal{D}A$$

is fully faithful. □

In [22], the category $\mathcal{C}_\Gamma$ is defined to be the smallest triangulated subcategory of $\mathcal{D}A$ containing the objects P_v for $v \in \Gamma$. By Proposition 2.1, this is equivalent to our definition of $\mathcal{C}_\Gamma$.

2.3.3. *Connection with the category of graded projective modules.* The category $\mathcal{C}_\Gamma$ is closely related to the homotopy category of complexes of graded projective A-modules studied in [17]. Let grmod- A be the abelian category of graded left A-modules that are finite dimensional as $\mathbf{k}$-vector spaces. Given a graded module M and an integer i, we denote by $M\{i\}$ the graded module obtained by shifting the grading by i, so that

$$M\{i\}_j = M_{j+i}.$$

Let $K(\text{grmod-}A)$ be the homotopy category of complexes of graded A-modules.

An object $X \subset K(\text{grmod-}A)$ has two gradings: one is the homological grading on the complex and the other is the grading internal to every term in the complex. As a result, we can view X as a bi-graded complex of $\mathbf{k}$ vector spaces

$$X = (X_{i,j})$$

such that i reflects the internal grading and j the homological grading. Then, for every j, the direct sum

$$\bigoplus_{i \in \mathbf{Z}} X_{i,j}$$

is an A-module. The differential on X has bi-degree $(0,1)$

$$\partial_{i,j} \colon X_{i,j} \to X_{i,j+1}.$$

Let $\overline{X}_*$ be the total complex associated to the bi-graded complex $X_{*,*}$. This is the complex whose terms are given by

$$\overline{X}_\ell = \bigoplus_{i+j=\ell} X_{i,j}$$

and where the differential $\overline{X}_\ell \to \overline{X}_{\ell+1}$ is a (signed) sum of the $\partial_{i,j}$. Then $\overline{X}$ is a dgm over A (considered as a dga with 0 differential). The procedure above yields an exact functor

$$\pi \colon K(\text{grmod-}A) \to K(\text{dgmod-}A).$$

Note that under this functor, the complex $X[1]$, obtained by homologically shifting X, and the complex $X\{-1\}$, obtained by shifting the internal grading of all terms of X, map to the same object. Thus, π collapses the homological and the internal grading to a single grading. It is easy to check that on the level of homs, the grading collapse leads to the following:

$$\operatorname{Hom}_{K(\text{dgmod-}A)}(\overline{X}, \overline{Y}) = \bigoplus_{i \in \mathbf{Z}} \operatorname{Hom}_{K(\text{grmod-}A)}(X, Y\{i\}[-i]). \tag{4}$$

Let $\widetilde{\mathcal{C}}_\Gamma \subset K(\text{grmod-}A)$ be the smallest triangulated subcategory containing $P_v\{i\}$ for all $v \in \Gamma$ and $i \in \mathbf{Z}$. (It is easy to check that that the $P_v\{i\}$ are the only indecomposable projective graded A-modules, so $\widetilde{\mathcal{C}}_\Gamma$ is just the homotopy category of complexes of projective graded A-modules.) Under the functor π, the category $\widetilde{\mathcal{C}}_\Gamma$ maps to the category $\mathcal{C}_\Gamma$:

$$\pi \colon \widetilde{\mathcal{C}}_\Gamma \to \mathcal{C}_\Gamma.$$

2.3.4. *Duality.* For $X, Y \in \widetilde{\mathcal{C}}_\Gamma$, let $\operatorname{Hom}^{i,j}(X,Y)$ be the space of morphisms of homological degree i and internal degree j; that is, set

$$\operatorname{Hom}^{i,j}(X,Y) = \operatorname{Hom}(X, Y\{i\}[j]).$$

The category $\widetilde{\mathcal{C}}_\Gamma$ enjoys the following duality.

Proposition 2.2. *For every $X, Y \in \widetilde{\mathcal{C}}_\Gamma$ we have a functorial isomorphism*

$$\operatorname{Hom}^{i,j}(X,Y) \cong \operatorname{Hom}^{2-i,-j}(Y,X)^\vee.$$

Proof. For P_v we have the trace map

$$\mathrm{tr}\colon \mathrm{Hom}^{2,0}(P_v, P_v) \to \mathbf{k}$$

that sends the loop map to 1. For any graded A-module M which is a direct sum of grading shifts of P_v, we can extend the trace map by taking the sum to get

$$\mathrm{tr}\colon \mathrm{Hom}^{2,0}(M, M) \to \mathbf{k}.$$

Furthermore, for any complex X whose terms X_i are direct sums of grading shifts of P_v, we can define a trace map

$$\mathrm{tr}\colon \mathrm{Hom}^{2,0}(X, X) \to \mathbf{k}$$

by setting

$$\mathrm{tr}(f) = \sum (-1)^i \,\mathrm{tr}(f_i\colon X_i \to X_i\{2\}).$$

It is easy to check that the trace map is functorial. It is also easy to check that for complexes X, Y whose terms are direct sums of shifts of P_v, the composite

$$\mathrm{Hom}^{i,j}(X, Y) \otimes \mathrm{Hom}^{2-i,-j}(Y, X) \xrightarrow{\text{composition}} \mathrm{Hom}^{2,0}(X, X) \xrightarrow{\mathrm{tr}} \mathbf{k}$$

is non-degenerate. The statement follows. □

Proposition 2.3. *The category $\mathcal{C}_\Gamma$ is 2-CY. That is, for every $X, Y \in \mathcal{C}_\Gamma$, we have an isomorphism*

$$\mathrm{Hom}^i(X, Y) \cong \mathrm{Hom}^{2-i}(Y, X)^\vee$$

functorial in X, Y.

Proof. Combine Proposition 2.2 and (4). □

Remark 2.4. Let $\{P_v \mid v \in \Gamma\}$ be a Γ-configuration of objects in a dg category, namely a collection of objects satisfying the Hom conditions as in (3). Then the endomorphism algebra $\mathrm{Hom}^*(\bigoplus P_v, \bigoplus P_v)$ is a dga whose cohomology algebra is the zig-zag algebra $A(\Gamma)$. If the endomorphism algebra is formal (quasi-isomorphic to its cohomology algebra), then the triangulated category generated by the objects P_v is equivalent to the category $\mathcal{C}_\Gamma$ (see [22, § 3]). In many cases, $A(\Gamma)$ is known to be intrinsically formal—any dga with cohomology $A(\Gamma)$ is formal. Suppose, for simplicity, that the ground field $\mathbf{k}$ has characteristic zero. Then $A(\Gamma)$ is intrinsically formal if Γ is a Dynkin graph of type A by [19] or of type D by [8] (see the discussion after Theorem 12), conjecturally also in type E [8, Conjecture 13], and also for any tree Γ of non-Dynkin type [8, Remark 14].

Remark 2.5. Let us relate $\mathcal{C}_\Gamma$ with the categories studied in [13] and [6]. Let $G \subset \mathrm{SL}_2(\mathbf{C})$ be a finite group and let $f\colon Y \to \mathbf{C}^2/G$ be the minimal resolution of singularities. Let $\mathcal{D} \subset \mathrm{Coh}(Y)$ be the full subcategory consisting of E such that $R\pi_* E = 0$. Let Γ be the dual graph of the exceptional divisor of f. The vertices of Γ correspond to the irreducible components of the exceptional divisor (which are copies of $\mathbf{P}^1$); the edges of Γ correspond to intersection points of the components. For $v \in \Gamma$, let $P_v \in \mathrm{Coh}(Y)$ be the sheaf $i_*\mathcal{O}(-1)$ where $i\colon \mathbf{P}^1 \to Y$ is the inclusion of the component of the exceptional divisor of f indexed by v. The objects P_v form a Γ-configuration and the category $\mathcal{D}$ is generated by them. Then the category $\mathcal{D}$ is equivalent to the category $\mathcal{C}_\Gamma$. This follows, for example, by Remark 2.4 in types A and D. It can also be shown directly in all types using [7].

Remark 2.6. In the categories $\mathcal{C}_\Gamma$, the spherical twists can be described as tensor products with certain complexes of bi-modules. This the point of view in [9].

2.3.5. *The standard t-structure.* The category $\widetilde{\mathcal{C}}_\Gamma$ admits a standard t-structure whose heart consists of so-called linear complexes [18, § 3]. A complex X of graded A-modules is called *linear* if for each i, the component X_i is a direct sum of modules of the form $P_v\{i\}$. For example, the following complex is linear

$$P_v \to P_w\{1\}.$$

The subcategory $\widetilde{\mathcal{C}}_\Gamma$ consisting of complexes homotopic to linear complexes forms the heart of a bounded t-structure. Likewise, its image in $\mathcal{C}_\Gamma$ also forms the heart of a bounded t-structure. Note that this image is the extension closure in $\mathcal{C}_\Gamma$ of the objects P_v for $v \in \Gamma$.

2.3.6. *The braid group action.* The Grothendieck group $K(\mathcal{C}_\Gamma)$ of $\mathcal{C}_\Gamma$ is the free abelian group generated by the classes of P_v for $v \in \Gamma$. The rule

$$\langle [X], [Y] \rangle = \sum (-1)^i \dim \operatorname{Hom}^i(X, Y)$$

defines a bilinear pairing on $K(\mathcal{C}_\Gamma)$. With this pairing, $K(\mathcal{C}_\Gamma)$ is isomorphic to the root lattice of Γ.

Recall that the Artin-Tits braid group B_Γ is the group generated by symbols σ_v for $v \in \Gamma$ modulo the following relations for every pair of vertices v, w:

$$\begin{aligned} \sigma_v\sigma_w\sigma_v &= \sigma_w\sigma_v\sigma_w \text{ if } v - w \text{ is an edge of } \Gamma, \\ \sigma_v\sigma_w &= \sigma_w\sigma_v \text{ otherwise.} \end{aligned} \tag{5}$$

The Coxeter group W_Γ is the quotient of B_Γ by the additional relations $\sigma_v^2 = 1$. (The generators of the Coxeter group are customarily denoted by roman symbols s_v.)

The Coxeter group W_Γ acts on the Grothendieck group $K(\mathcal{C}_\Gamma)$. The generator s_v acts by reflection in the hyperplane perpendicular to the class of P_v

$$s_v \colon x \mapsto x - \langle x, P_v \rangle [P_v].$$

The action of W_Γ on $K(\mathcal{C}_\Gamma)$ lifts to an action of B_Γ on $\mathcal{C}_\Gamma$. The generator σ_v acts by the spherical twist in P_v:

$$\sigma_v \colon X \mapsto \sigma_{P_v} X.$$

3. Phase spread reduction using spherical twists

The goal of this section is to prove that by applying suitable spherical twists, we can predictably increase/decrease the bottom/top phase of an object. Throughout, fix a **k**-linear 2-CY triangulated category $\mathcal{C}$ of finite type and a dg enhancement. Also fix a stability condition τ on $\mathcal{C}$.

The following is standard.

Lemma 3.1 (Sandwich lemma). *Let $X \to Y \to Z \xrightarrow{+1}$ be an exact triangle. Then*

$$\begin{aligned} &\phi^-(Y) \geq \min\{\phi^-(X), \phi^-(Z)\}, \text{ and} \\ &\phi^+(Y) \leq \max\{\phi^+(X), \phi^+(Z)\}. \end{aligned}$$

Proof. We prove the first inequality; the second is analogous. Recall that $\lfloor Y \rfloor$ is the HN piece of Y of smallest phase, and this smallest phase is denoted by $\phi^-(Y)$. We have a nonzero map $Y \to \lfloor Y \rfloor$. As a result, there is either a nonzero map $X \to \lfloor Y \rfloor$ or a nonzero map $Z \to \lfloor Y \rfloor$. This shows that $\phi^-(Y) \geq \phi^-(X)$ or $\phi^-(Y) \geq \phi^-(Z)$. Equivalently, $\phi^-(Y) \geq \min\{\phi^-(X), \phi^-(Z)\}$. □

We now investigate the effect of applying suitable spherical twists on the bottom and the top phase.

Proposition 3.2. *Let X be a spherical stable object of $\mathcal{C}$ such that X is the unique (up to isomorphism) stable object of its phase. Let Y be any object of $\mathcal{C}$. We have the following.*

1. *If $\phi(X) \leq \phi^-(Y)$, then $\phi(X) < \phi^-(\sigma_X^{-1}(Y))$.*
2. *If $\phi^+(Y) \leq \phi(X)$, then $\phi^+(\sigma_X(Y)) < \phi(X)$.*

Proof. We prove the first inequality; the second is analogous. The key is the exact triangle

$$X \otimes \operatorname{Hom}^*(Y, X)^\vee[-1] \to \sigma_X^{-1}(Y) \to Y \xrightarrow{+1}, \tag{6}$$

obtained by rotating the triangle in (2) in § 2.

We first prove the proposition assuming the strict inequality

$$\phi(X) < \phi^-(Y). \tag{7}$$

Let $V = \operatorname{Hom}^*(Y, X)^\vee[-1]$. Then V is a direct sum of shifts of copies of $\mathbf{k}$. The inequality (7) implies that $\operatorname{Hom}^i(Y, X) = 0$ for $i \leq 0$. Therefore, V is a direct sum of copies of $\mathbf{k}[j]$ for $j \geq 0$. As a result, we have

$$\phi(X) \leq \phi^-(X \otimes V). \tag{8}$$

Thanks to (7) and (8), we can apply the sandwich lemma (Lemma 3.1) to the key triangle (6) to get

$$\phi(X) \leq \phi^-(\sigma_X^{-1}(Y)).$$

To show that the inequality is strict, it suffices to show that $\sigma_X^{-1}Y$ does not have a nonzero map to any stable object of phase $\phi(X)$. By our assumption, the only such stable object is X itself. Consider a map $f\colon \sigma_X^{-1}(Y) \to X$. Applying σ_X gives a map $\sigma_X(f)\colon Y \to X[-1]$. Since $\phi(X[-1]) < \phi(X) < \phi^-(Y)$, the map $\sigma_X(f)$ must be zero. Therefore f is zero. The proof is thus complete, assuming $\phi(X) < \phi^-(Y)$.

We now treat the case $\phi(X) = \phi^-(Y)$. Recall that we have assumed that X is the only stable object of its phase. Furthermore, X is spherical, so it has no self-extensions. Therefore, $\lfloor Y \rfloor$ is a direct sum of copies of X. This means that we have an exact triangle

$$Z \to Y \to X^{\oplus n} \xrightarrow{+1}, \tag{9}$$

where $\phi(X) < \phi^-(Z)$. The previous argument now applies to Z, and we get $\phi(X) < \phi^{-1}\left(\sigma_X^{-1}Z\right)$. Applying σ_X^{-1} to the triangle in (9) gives the triangle

$$\sigma_X^{-1}(Z) \to \sigma_X^{-1}(Y) \to X[1]^{\oplus n} \xrightarrow{+1}. \tag{10}$$

By applying the sandwich lemma (Lemma 3.1) to (10), we conclude that $\phi(X) < \phi^-(\sigma_X^{-1}(Y))$. ∎

The following shows that the improvement on one end achieved by Proposition 3.2 does not cause a deterioration at the other end, assuming that the object Y has a sufficiently large phase spread.

Proposition 3.3. *Let X be a spherical stable object of $\mathcal{C}$. Assume that X is the unique stable object of its phase. Let Y be any object of $\mathcal{C}$ such that $\operatorname{Hom}^i(Y, Y) = 0$ for any $i < 0$ and $\phi^+(Y) - \phi^-(Y) \geq 1$. The following hold.*

1. *If $\phi^-(Y) = \phi(X)$, then $\phi^+(Y) \geq \phi^+(\sigma_X^{-1}(Y))$.*
2. *If $\phi^+(Y) = \phi(X)$, then $\phi^-(Y) \leq \phi^-(\sigma_X(Y))$.*

Proof. We prove the first statement; the second is analogous. Again, the key is the exact triangle

$$X \otimes \mathrm{Hom}^*(Y, X)^\vee[-1] \to \sigma_X^{-1}(Y) \to Y \xrightarrow{+1}, \tag{11}$$

obtained by rotating the triangle (2) in § 2. Let $V = \mathrm{Hom}^*(Y, X)^\vee[-1]$. By the sandwich lemma (Lemma 3.1), it suffices to show that

$$\phi^+(Y) \geq \phi^+(X \otimes V). \tag{12}$$

Let us compute $\phi^+(X \otimes V)$. Let ℓ be the largest integer such that $\mathrm{Hom}^\ell(Y, X) \neq 0$. Then V is a direct sum of copies of $\mathbf{k}[j]$ with $j \leq \ell - 1$, and including at least one copy of $\mathbf{k}[\ell - 1]$. Therefore, we get

$$\phi^+(X \otimes V) = \phi(X[\ell - 1]). \tag{13}$$

Thus, showing (12) is equivalent to showing

$$\phi^+(Y) \geq \phi(X[\ell - 1]).$$

First suppose $\ell \leq 2$. Then we have

$$\phi^-(Y) + 1 = \phi(X[1]) \geq \phi(X[\ell - 1]).$$

Since $\phi^+(Y) - \phi^-(Y) \geq 1$, we conclude that

$$\phi^+(Y) \geq \phi(X[\ell - 1]),$$

as desired.

Next suppose $\ell > 2$. Then the 2-CY property implies that

$$\mathrm{Hom}^{2-\ell}(X, Y) \neq 0.$$

Since we have assumed that X is the only stable object of its phase, and X has no self-extensions, and $\phi^-(Y) = \phi(X)$, it follows that $\lfloor Y \rfloor$ is a direct sum of copies of X. Therefore, we also have

$$\mathrm{Hom}^{2-\ell}(\lfloor Y \rfloor, Y) \neq 0.$$

Define the object K by the following exact triangle:

$$K \to Y \to \lfloor Y \rfloor \xrightarrow{+1} .$$

Let $f \in \mathrm{Hom}^{2-\ell}(\lfloor Y \rfloor, Y)$ be a non-zero element. The composition of $Y \to \lfloor Y \rfloor$ with f gives a map $Y \to Y[2 - \ell]$. Since this is a map of negative degree from Y to itself, it must be zero. Therefore f factors as the composite of $\lfloor Y \rfloor \to K[1]$ with a (non-zero) map $g \colon K[1] \to Y[2 - \ell]$

$$\begin{tikzcd}
Y \arrow[r] \arrow[dr, "0"'] & \lfloor Y \rfloor \arrow[r] \arrow[d, "f"] & K[1] \arrow[r, "+1"] \arrow[dl, "g"] & {} \\
 & Y[2-\ell]. & &
\end{tikzcd}$$

Since g is non-zero, we get

$$\phi^+(Y[2 - \ell]) \geq \phi^-(K[1]).$$

By construction, we have

$$\phi^-(K) > \phi^-(Y) = \phi(X).$$

By combining the last two inequalities, we see that

$$\phi^+(Y[2 - \ell]) > \phi(X[1]).$$

Therefore, we get

$$\phi^+(Y) > \phi(X[\ell-1]),$$

as desired. □

The following is an analogue of Proposition 3.3 for Y of small spread.

Proposition 3.4. *Let X be a spherical stable object of $\mathcal{C}$. Assume that X is the unique stable object of its phase. Let Y be any object of $\mathcal{C}$ such that $0 < \phi^+(Y) - \phi^-(Y) < 1$. Assume, furthermore, that $\mathrm{Hom}^0(Y,Y) = \mathbf{k}$. Then the following hold.*

(1) If $\phi^-(Y) = \phi(X)$, then $\phi^+(Y) \geq \phi^+(\sigma_X^{-1}(Y))$.
(2) If $\phi^+(Y) = \phi(X)$, then $\phi^-(Y) \leq \phi^-(\sigma_X(Y))$.

Proof. We prove the first statement; the second is analogous. We begin as in the proof of Proposition 3.3. Consider the triangle

$$X \otimes \mathrm{Hom}^*(Y,X)^\vee[-1] \to \sigma_X^{-1}(Y) \to Y \xrightarrow{+1} . \tag{14}$$

Let $V = \mathrm{Hom}^*(Y,X)^\vee[-1]$. It suffices to show that

$$\phi^+(Y) \geq \phi^+(X \otimes V).$$

Let ℓ be the largest integer such that $\mathrm{Hom}^\ell(Y,X) \neq 0$. Then we must show that

$$\phi^+(Y) \geq \phi(X[\ell-1]).$$

By the 2-CY property, we have $\mathrm{Hom}^\ell(Y,X) \cong \mathrm{Hom}^{2-\ell}(X,Y)^\vee$. Since

$$\phi^+(Y) < \phi^-(Y) + 1 = \phi(X) + 1,$$

there cannot be a non-zero map from X to $Y[k]$ for $k < 0$. As a result, we must have $\ell \leq 2$.

First suppose $\ell \leq 1$. Then we have

$$\phi^+(Y) \geq \phi^-(Y) = \phi(X) \geq \phi(X[\ell-1]),$$

as desired.

We rule out $\ell = 2$. Let us show that if $\ell = 2$, then Y is isomorphic to X, which contradicts the assumption that

$$0 < \phi^+(Y) - \phi^{-1}(Y).$$

Let $\mathcal{P}$ be the slicing defined by the stability condition τ. Since $\phi^+(Y) - \phi^-(Y) < 1$ and $\phi(X) = \phi^-(Y)$, both X and Y lie in the abelian category $\mathcal{P}[\alpha, \alpha+1)$ for $\alpha = \phi^-(Y)$. Since $\phi^-(Y) = \phi(X)$ and X is the only stable object of its phase and X has no self-extensions, $\lfloor Y \rfloor$ is a direct sum of copies of X. Say $\lfloor Y \rfloor = X^{\oplus n}$.

If $\ell = 2$, we have a non-zero map $i \colon X \to Y$. Consider the composite

$$X \xrightarrow{i} Y \to \lfloor Y \rfloor. \tag{15}$$

We show that the composite is non-zero. Equivalently, we must show that i does not factor through the kernel K of $Y \to \lfloor Y \rfloor$. In fact, let us prove that there are no non-zero maps from X to K.

Since $\mathrm{Hom}^0(Y,Y) = \mathbf{k}$, every non-zero map from Y to itself is an isomorphism. A non-zero map $X \to K$ gives a non-zero map $Y \to Y$ that is not an isomorphism, namely the composite

$$Y \twoheadrightarrow \lfloor Y \rfloor = X^{\oplus n} \twoheadrightarrow X \to K \hookrightarrow Y.$$

Therefore, there are no non-zero maps $X \to K$.

Since we have $\lfloor Y \rfloor = X^{\oplus n}$ and the composite in (15) is non-zero, there is a map $\pi \colon Y \to X$ such that $\pi \circ i \colon X \to X$ is non-zero. But X is spherical, so $\pi \circ i$ must be an isomorphism.

That is, X is a direct summand of Y. Since $\mathrm{Hom}(Y,Y) = \mathbf{k}$, there are no non-trivial direct summands of Y, and hence $\pi \colon Y \to X$ must be an isomorphism. □

The following theorem combines the results above. Recall that $\mathcal{C}$ is a $\mathbf{k}$-linear 2-CY triangulated category of finite type with a fixed dg enhancement and τ is a stability condition on $\mathcal{C}$.

Theorem 3.5. *Let X be a τ-stable spherical object of $\mathcal{C}$. Assume that X is the unique stable object of its phase. Let Y be an object of $\mathcal{C}$ such that*

$$\mathrm{Hom}^i(Y,Y) = \begin{cases} \mathbf{k} & \text{if } i = 0 \\ 0 & \text{if } i < 0. \end{cases}$$

Suppose

(1) $\phi(X) = \phi^-(Y)$, in which case set $Y' = \sigma_X^{-1}(Y)$; or

(2) $\phi(X) = \phi^+(Y)$, in which case set $Y' = \sigma_X(Y)$.

If $\phi^+(Y) - \phi^-(Y) > 0$, then

$$\phi^+(Y') - \phi^-(Y') < \phi^+(Y) - \phi^-(Y).$$

Proof. Let us prove the first statement; the second is analogous. By Proposition 3.2, we get

$$\phi^-(Y) < \phi^-(Y').$$

If $\phi^+(Y) - \phi^-(Y) \geq 1$, we apply Proposition 3.3, and if $\phi^+(Y) - \phi^-(Y) < 1$, we apply Proposition 3.4 and get

$$\phi^+(Y) \geq \phi^+(Y').$$

The theorem follows. □

Remark 3.6. The uniqueness assumption on X in Proposition 3.2, Proposition 3.3, Proposition 3.4, and Theorem 3.5 can be weakened if we know more about the stable factors of Y and its twists. For example, if Y is a direct sum of spherical objects, then the stable factors of Y and its twists are also spherical [11, Corollary 2.3]. For such a Y, the statements and proofs of these results hold under the weaker assumption that X is the unique spherical stable object of its phase.

4. Spherical stable objects in quiver categories

In this section, fix Γ to be any connected finite graph without loops or multiple edges. Let $\mathcal{C} = \mathcal{C}_\Gamma$ be the 2-CY category associated to Γ as defined in § 2.3.

Recall that the Grothendieck group $K(\mathcal{C})$ with the Hom pairing is naturally identified with the root lattice of Γ. Since $\dim(\mathrm{Hom}^*(X,X)) = 2$ for any spherical object X, its class $[X]$ in $K(\mathcal{C})$ is a real root. We refer the reader to [3] [10] for the theory of Coxeter groups and their root lattices.

Let τ be a stability condition on $\mathcal{C}$ with central charge $Z \colon K(\mathcal{C}) \to \mathbf{C}$ and slicing $\mathcal{P}$.

Proposition 4.1. *Assume that τ is generic in the following sense: Z maps distinct real roots to complex numbers of distinct phase. Suppose X is a τ-semistable spherical object. Then X is τ-stable, and it is the unique τ-stable spherical object of phase $\phi = \phi(X)$.*

Proof. To show that X is stable, we must show that X is simple in the abelian category $\mathcal{P}(\phi)$. Let $S \subset X$ be a non-zero simple sub-object. By [11, Corollary 2.3], S must be spherical. Then the class of S in $K(\mathcal{C})$ is a real root. Since $Z(S)$ has the same argument as

$Z(X)$, the genericity assumption on τ means that $X = S$ in $K(\mathcal{C})$. But then $X/S = 0$ in $K(\mathcal{C})$, and since X/S is in a heart of τ, this implies that $X = S$.

Next, suppose Y is another τ-stable spherical object of the same phase as X. Again by the genericity of τ, we have $X = Y$ in $K(\mathcal{C})$. But then we get

$$\begin{aligned}\langle X, Y\rangle &= \dim \operatorname{Hom}^0(X, Y) - \dim \operatorname{Hom}^1(X, Y) + \dim \operatorname{Hom}^2(X, Y)\\ &= \langle X, X\rangle = 2,\end{aligned}$$

which implies $\operatorname{Hom}^0(X, Y) \neq 0$ by the 2-CY property. Since X and Y are simple objects of $\mathcal{P}(\phi)$, this forces $X \cong Y$. □

For $v \in \Gamma$, we have the spherical object $P_v \in \mathcal{C}$. The extension closure of these objects is the heart of a bounded t-structure on $\mathcal{C}$. We call this the *standard heart* and denote it $\heartsuit$. The objects P_v are simple in $\heartsuit$. We say that a stability condition is *standard* if its $[0, 1)$ heart is $\heartsuit$. We now give an effective construction of a τ-stable spherical object of every possible class, for a generic standard stability condition τ.

Let w be an arbitrary positive root. Write

$$w = s_{v_n} \cdots s_{v_1} v, \tag{16}$$

where v is a simple root and $s_{v_1}, \dots, s_{v_n}$ are reflections in the simple roots $v_1, \dots, v_n$. Set $v_0 = v$. Associate to (16) the *root sequence* $R_0, \dots, R_n$ defined by

$$R_i = s_{v_n} \cdots s_{v_{i+1}}(v_i).$$

Note that all R_i are roots and $R_0 = w$. Furthermore, if (16) is a minimal expression for w, then the roots R_i are distinct positive roots [10, § 5.6, Exercise 1].

Let $\epsilon_1, \dots, \epsilon_n$ be ± 1. Consider an object X of $\mathcal{C}$ defined by

$$X = \sigma_{v_n}^{\epsilon_n} \circ \cdots \circ \sigma_{v_1}^{\epsilon_1}(P_v), \tag{17}$$

where σ_{v_i} is the spherical twist in P_{v_i}. The ϵ's allow us to divide the root sequence into sub-sequences corresponding to positive and negative exponents:

$$\begin{aligned}R_+ &= \{R_i \mid \epsilon_i = 1\},\\ R_- &= \{R_i \mid \epsilon_i = -1\}.\end{aligned}$$

We call R_0 the *neutral root*.

Let τ be a stability condition on $\mathcal{C}$ such that the $[0, 1)$ heart of τ is the standard heart. Let Z be the central charge of τ. Let $\mathbf{H} \subset \mathcal{C}$ be the half-open upper half plane:

$$\mathbf{H} = \{z \mid \Im(z) > 0\} \cup \mathbf{R}_{>0}.$$

Let $\alpha = Z(R_0)$. Since R_0 is a positive root, $Z(R_0)$ lies in $\mathbf{H}$. It divides $\mathbf{H}$ into two pieces

$$\begin{aligned}\mathbf{H}_+ &= \{z \mid \arg z > \arg \alpha\},\\ \mathbf{H}_- &= \{z \mid \arg z < \arg \alpha\},\end{aligned} \tag{18}$$

where arg is taken in $[0, \pi)$.

Figure 1 shows an example of the construction above for a stability condition on the A_3-category.

Since all the roots in the root sequence R of X are positive roots, they are mapped to the upper half plane $\mathbf{H}$ by the central charge. The semi-stability of X depends on their position with respect to R_0.

Proposition 4.2. *With the notation above, the object X defined by* (17) *is τ-semistable if and only if $Z(R_+) \subset \mathbf{H}_+$ and $Z(R_-) \subset \mathbf{H}_-$.*

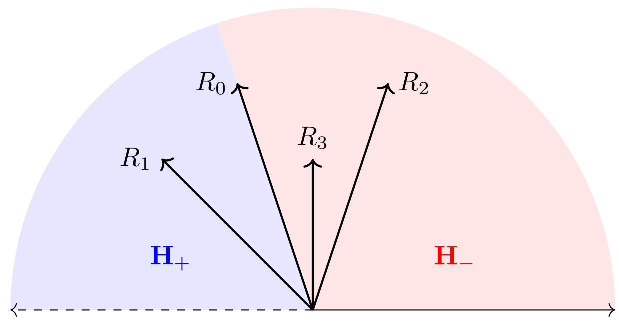

FIGURE 1. Consider the A_3 quiver with simple roots α_i and simple reflections s_i. Consider the root sequence for $w = s_2 s_3 s_1(\alpha_2)$. The central charge chosen for the diagram above maps R_1 to $\mathbf{H}_+$ and R_2, R_3 to $\mathbf{H}_-$. By Proposition 4.2, the stable object of class w is $\sigma_2^{-1}\sigma_3^{-1}\sigma_1(P_2)$.

Let τ be a generic standard stability condition. Proposition 4.2 gives an effective construction of the unique τ-stable object of class $w = R_0$, as follows (see Figure 1).

(1) Write a minimal expression $w = s_{v_n} \cdots s_{v_1} v$.
(2) Construct the root sequence $R = (R_0, \dots, R_n)$ by setting $R_i = s_{v_n} \cdots s_{v_{i+1}}(v_i)$, with $v_0 = v$.
(3) Since the R_i are positive roots, Z maps them to the upper half plane $\mathbf{H}$. Decompose $\mathbf{H}$ into $\mathbf{H}_+$ and $\mathbf{H}_-$ as in (18) and decompose R into R_+ and R_- as

$$R_+ = \{R_i \mid Z(R_i) \in \mathbf{H}_+\},$$
$$R_- = \{R_i \mid Z(R_i) \in \mathbf{H}_-\}.$$

(4) Set $\epsilon_i = +1$ (resp -1) if $R_i \in R_+$ (resp R_-), and let

$$X = \sigma_{v_n}^{\epsilon_n} \circ \cdots \circ \sigma_{v_1}^{\epsilon_1}(P_v).$$

Then, by Proposition 4.2 X is a τ-semi-stable object of class w. By Proposition 4.1, it is in fact the unique stable object of class w.

We need some preparation to prove Proposition 4.2. Let $\heartsuit \subset \mathcal{C}$ be the standard heart. Set

$$K = K(\heartsuit) = K(\mathcal{C}).$$

Denote by $[X]$, the class in K of an object X.

Lemma 4.3. *Let $v \in \Gamma$ and $X \in \heartsuit$ be any object. The twist $\sigma_{P_v}^{-1} X$ lies in $\heartsuit$ if and only if P_v is not a sub-object of X. Similarly, the twist $\sigma_{P_v} X$ lies in $\heartsuit$ if and only if P_v is not a quotient of X.*

Proof. We prove the first statement; the second is analogous. Set $P = P_v$. We have the exact triangle

$$P \otimes \mathrm{Hom}^*(X, P)^\vee[-1] \to \sigma_P^{-1}(X) \to X \xrightarrow{+1} . \tag{19}$$

Since both P and X lie in $\heartsuit$, the 2-CY property implies that $\mathrm{Hom}^i(X, P)$ is zero for $i < 0$ and $i > 2$.

Assume that P is not a sub of X in $\heartsuit$. Since P is simple in $\heartsuit$, we must have $\mathrm{Hom}^0(P, X) = 0$. By the 2-CY property, this implies $\mathrm{Hom}^2(X, P) = 0$.

Let $V = \operatorname{Hom}^*(X,P)^\vee[-1]$. Then V is a direct sum of shifted copies of $\mathbf{k}$. Since $\operatorname{Hom}^i(X,P) = 0$ if $i \notin \{0,1\}$, the complex V must be a direct sum of copies of $\mathbf{k}[j]$ for $j = -1, 0$.

In the exact triangle (19), the two extreme terms lie in $\mathcal{P}[-1,1)$. As a result, the truncation $\sigma_P^{-1}(X)_{<0}$ also lies in $\mathcal{P}[-1,1)$. We must show that it lies in $\heartsuit = \mathcal{P}[0,1)$. That is, we must show that its truncation to $\mathcal{P}(-\infty,0)$ is zero.

Note that the truncation $\sigma_P^{-1}(X)_{<0}$ lies in $\heartsuit[-1]$ and coincides with $H^1(\sigma_P^{-1}(X))[-1]$. Since $H^1(X) = 0$, the cohomology long exact sequence applied to the triangle (19) shows that $\sigma_P^{-1}(X)_{<0}$ is a quotient of a direct sum of copies of $P[-1]$ in $\heartsuit[-1]$. Since P is simple in $\heartsuit$, the object $\sigma_P^{-1}(X)_{<0}$ must itself be a direct sum of copies of $P[-1]$.

If $\sigma_P^{-1}(X)_{<0}$ were non-zero, then we would have a non-zero map $\sigma_P^{-1}(X)_{<0} \to P[-1]$, and hence a non-zero map $\sigma_P^{-1}(X) \to P[-1]$. By applying σ_P, we would then obtain a non-zero map $X \to P[-2]$, which is a contradiction. We conclude that $\sigma_P^{-1}X$ lies in $\heartsuit$.

Conversely, if P is a sub of X, then we have a non-zero map $P \to X$ and hence a non-zero map $\sigma_P^{-1}P = P[1] \to \sigma_P^{-1}X$. It follows that $\sigma_P^{-1}X$ is not in $\heartsuit$. □

Consider an object $X \in \heartsuit$. We say that a subset $S \subset K$ *envelops* the subs (resp. quotients) of X if for every sub (resp. quotient) object Y of X, the class $[Y]$ can be expressed as non-negative linear combination of the elements of S and $\pm[X]$. Observe that if S envelops the subs of X then $-S$ envelops the quotients of X, and vice-versa.

Lemma 4.4. *Let X be an object of $\heartsuit$ and let $v \in \Gamma$. Let S be a subset of K and set $S' = s_v(S) \cup \{v\}$.*

1. *If S envelops the subs of X and $\sigma_{P_v}^{-1}X$ lies in $\heartsuit$, then S' envelops the subs of $\sigma_{P_v}^{-1}X$.*
2. *If $S \subset K$ envelops the quotients of X and $\sigma_{P_v}X$ lies in $\heartsuit$, then S' envelops the quotients of $\sigma_{P_v}X$.*

Proof. We prove the first assertion; the second is similar.

Set $P = P_v$. Let Y be any sub of $\sigma_P^{-1}X$. We must prove that $[Y]$ is a non-negative linear combination of the elements of S' and $\pm[\sigma_P^{-1}(X)]$.

First suppose P is not a quotient of Y. Set

$$Q = \operatorname{coker}(Y \to \sigma_P^{-1}X).$$

Since $\sigma_P\sigma_P^{-1}X$ lies in $\heartsuit$, by Lemma 4.3, P is not a quotient of $\sigma_P^{-1}X$. Therefore P is not a quotient of Q. By applying σ_P to the exact sequence

$$0 \to Y \to \sigma_P^{-1}X \to Q \to 0,$$

we get an exact triangle

$$\sigma_P Y \to X \to \sigma_P Q \xrightarrow{+1},$$

whose terms are in $\heartsuit$ by Lemma 4.3. Therefore, it is an exact sequence in $\heartsuit$. Since S envelops the subs of X, the class $[\sigma_P Y] = s_v[Y]$ is a non-negative linear combination of elements of S and $\pm[X]$. Equivalently, the class $[Y]$ is a non-negative linear combination of the elements of $s_v(S)$ and $\pm[\sigma_P^{-1}X]$.

It remains to treat the case when P is a quotient of Y. Define $Y' \subset Y$ be such that we have an exact sequence

$$0 \to Y' \to Y \to P^{\oplus n} \to 0$$

for some n and P is not a quotient of Y (such a Y' exists because $\heartsuit$ is a finite-length category). By the argument above, $[Y']$ is a non-negative linear combination of the elements of $s_v(S)$ and $\pm[\sigma_P^{-1}X]$. But then $[Y]$ is a non-negative linear combination of the elements of $s_v(S) \cup \{v\}$ and $\pm[\sigma_P^{-1}X]$, as desired. □

Consider an object X of $\mathcal{C}$ defined as in (17):

$$X = \sigma_{v_n}^{\epsilon_n} \circ \cdots \circ \sigma_{v_1}^{\epsilon_1}(P_v),$$

and the root sequence R_i, divided into a positive sub-sequence R_+, a negative sub-sequence $R-$, and the neutral root R_0.

Lemma 4.5. *In the above setup, suppose there exists a linear functional $\lambda\colon K(\mathcal{C}) \to \mathbf{R}$ such that $\lambda(R_0) = 0$, and $\lambda(R_+) \subset \mathbf{R}_{>0}$, and $\lambda(R_-) \subset \mathbf{R}_{<0}$. Then X lies in the heart $\heartsuit$. Furthermore, the set $S_- = R_- \cup -R_+$ (resp. $S_+ = R_+ \cup -R_-$) envelops the subs (resp. quotients) of X.*

Corrigendum. In the published version of the paper, the lemma above says that R_- (resp. R_+) envelops the subs (resp. quotients) of X. That statement is incorrect. This version corrects the statement of Lemma 4.5, its proof, and also makes the small changes necessary in the proof of Proposition 4.2.

Proof. We induct on n. If $n = 0$, then $X = P_v$ is simple, both R_+ and R_- are empty, and the statement holds.

Assume the statement for $(n-1)$. Let

$$X' = \sigma_{v_{n-1}}^{\epsilon_{n-1}} \circ \cdots \circ \sigma_{v_1}^{\epsilon_1}(P_v),$$

and let R' denote the root sequence for X'. Then we have $R'_i = s_{v_n} R_i$ for $i = 0, \ldots, n-1$. Note that

$$\lambda' = \lambda \circ s_{v_n} : K(\mathcal{C}) \to \mathbf{R}$$

is a linear functional that vanishes on the neutral root R'_0 for X' and takes opposite signs on the positive and the negative sub-sequences R'_+ and R'_-. By the induction hypothesis, X' lies in the heart $\heartsuit$ and its subs (resp. quotients) are enveloped by $S'_- = R'_- \cup -R'_+$ (resp. $S'_+ = R'_+ \cup -R'_-$).

Observe that $R_n = v_n$. Suppose $\epsilon_n = -1$. Then $R_n \in R_-$. Since λ is negative on R_-, it is positive on $-R_n = s_{v_n}(R_n)$. By construction, λ' is negative on R'_-, positive on R_n, and zero on $[X']$. So $R_n = [P_{v_n}]$ cannot be a positive linear combination of elements of R'_- together with $\pm[X']$.

Since S'_- envelops the subs of X', we conclude that P_{v_n} is not a sub of X'. Hence, by Lemma 4.3, X is in the heart and its subs are enveloped by $s_1(S'_-) \cup \{v_n\} = S_-$. The proof when $\epsilon_n = +1$ is similar. □

We now have the tools to finish the proof of Proposition 4.2.

Proof of Proposition 4.2. Once again, set

$$X = \sigma_{v_n}^{\epsilon_n} \circ \cdots \circ \sigma_{v_1}^{\epsilon_1}(P_v),$$

with S_+ and S_- defined as in Lemma 4.5. Suppose Z maps R_+ and R_- to $\mathbf{H}_+$ and $\mathbf{H}_-$, respectively. Choose a linear functional $\ell\colon \mathbf{C} \to \mathbf{R}$ that vanishes on $\alpha = Z(R_0)$ and takes positive (resp. negative) values on $\mathbf{H}_+$ (resp. $\mathbf{H}_-$). Set $\lambda = \ell \circ Z$. Then λ satisfies the hypotheses of Lemma 4.5. As a result, X is in the heart $\heartsuit$.

To show that X is semi-stable, consider a sub $Y \subset X$. But S_- envelops the subs of X, that is, $[Y]$ is a non-negative linear combination of elements of S_- and $\pm[X]$. By applying Z, we obtain that $Z(Y)$ is a non-negative linear combination of elements of $Z(S_-)$ and $\pm\alpha$.

Note that $Z(S_-) \subset \mathbf{H}_- \cup -\mathbf{H}_+$. Since we know that $Z(Y) \subset \mathbf{H}$, we conclude that $Z(Y)$ lies in $\mathbf{H}_- \cup \mathbf{R}_{>0} \cdot \alpha$. As a result, we have $\arg Y \leq \arg X$. Since this is true for any sub $Y \subset X$, we conclude that X is semi-stable. □

5. Applications

In this section, we reap the benefits of the results proved in the previous sections.

Fix the following notation:

- $\mathcal{C}$ a $\mathbf{k}$-linear 2-CY triangulated category of finite type with a fixed dg enhancement,
- τ a stability condition on $\mathcal{C}$,
- Φ_τ the subset of $\mathbf{R}$ consisting of the phases of τ-stable spherical objects
- G_τ the group of auto-equivalences of $\mathcal{C}$ generated by the twists in τ-stable spherical objects.

Proposition 5.1. *Assume that τ admits at most one spherical stable object of every phase and $\Phi_\tau \subset \mathbf{R}$ is discrete. Then every spherical object in $\mathcal{C}$ is in the G_τ-orbit of a τ-stable spherical object.*

Proof. Let Y be any spherical object of $\mathcal{C}$. We denote by $|Y|$ the *spread* of Y, which is the quantity

$$|Y| = \phi^+(Y) - \phi^-(Y).$$

Note that since the stable factors of the HN pieces of Y must be spherical [11, Corollary 2.3], both $\phi^+(Y)$ and $\phi^-(Y)$ lie in Φ_τ, and hence their difference lies in the set $\{a - b \mid a, b \in \Phi, a \geq b\}$, which is a discrete subset of $\mathbf{R}$.

We induct on $|Y|$. If $|Y| = 0$, then Y is τ-stable, and we are done.

Since Y is spherical, the stable factors of its HN pieces are also spherical [11, Corollary 2.3]. In particular, there is a spherical stable object of phase $\phi^-(Y)$, and it is unique by our assumption; call it X. Consider $Y' = \sigma_X^{-1} Y$. By Theorem 3.5, we have $|Y'| < |Y|$. By the induction hypothesis, Y' lies in the G_τ orbit of P, and hence so does Y.

Note that the same argument works with $Y' = \sigma_Z Y$ where Z is the unique τ-stable spherical object of phase $\phi^+(Y)$. □

Now let $\mathcal{C} = \mathcal{C}_\Gamma$ (see § 2.3 for the definition). Recall that the braid group of Γ acts on $\mathcal{C}$ by spherical twists.

Corollary 5.2 (Theorem 1.1). *Let $\mathcal{C}$ be the 2-CY category associated to a quiver of finite (ADE) type. Then every spherical object of $\mathcal{C}$ is in the braid group orbit of a simple object of the standard heart.*

Proof. Choose a generic standard stability condition τ on $\mathcal{C}$ with central charge $Z\colon K(\mathcal{C}) \to \mathbf{C}$. In particular, assume that Z maps distinct roots to complex numbers of distinct arguments. Then, by Proposition 4.1, there is at most one spherical stable object of every phase.

Let $\Phi_\tau \subset \mathbf{R}$ be the set of phases of spherical stable objects. Since the class of a spherical object in $K(\mathcal{C})$ is a root, of which there are only finitely many, the set Φ_τ consists of integer translates of a finite set. In particular, Φ_τ is discrete.

The simple objects of the standard heart are τ-stable and spherical. So the image of the braid group lies in G_τ. It is not hard to see that the image is, in fact, G_τ. To see this, let X be a τ-stable spherical object. We must show that σ_X lies in the image of the braid group.

From Proposition 4.2, we know that $X = \beta Y$, where Y is a simple object in the standard heart, and β is in the image of the braid group. Then

$$\sigma_X = \beta \sigma_Y \beta^{-1}$$

is also in the image of the braid group.

We now apply Proposition 5.1 and conclude that every spherical object is in the braid group orbit of a τ-stable spherical object. But we already know that every τ-stable spherical object is in the braid group orbit of a simple object of the standard heart. The proof is now complete. □

Remark 5.3 (Choice of writing). Note that in Proposition 5.1, we have a choice of applying a positive or a negative twist, leading to different expressions for a given spherical object as a braid image of a simple object. It is an interesting problem to understand these different expressions.

Proposition 5.4. *Assume that τ admits at most one spherical stable object of every phase and $\Phi_\tau \subset \mathbf{R}$ is discrete. Let Y be a direct sum of spherical objects of $\mathcal{C}$ such that $\operatorname{Hom}^i(Y, Y) = 0$ for $i < 0$. Then there exists a stability condition ω in the G_τ-orbit of τ such that Y lies in the $[\alpha, \alpha + 1)$-heart of ω for some α.*

Proof. We induct on the spread $|Y| = |Y|_\tau$. Note that this quantity lies in the discrete set $\{a - b \mid a, b \in \Phi_\tau\}$.

If $|Y|_\tau < 1$, then we are done. Simply take $\omega = \tau$ and $\alpha = \phi^-(Y)$.

Suppose $|Y|_\tau \geq 1$. Let X be the unique spherical τ-stable object of phase $\phi^-(Y)$. Let $\tau' = \sigma_X \tau$. Note that the group G_τ and the set Φ_τ are unchanged if we replace τ by τ'. By Proposition 3.2 and Proposition 3.3, we have

$$|Y|_{\tau'} = |\sigma_X^{-1} Y|_\tau < |Y|_\tau.$$

We conclude the result by the induction hypothesis. □

Corollary 5.5 (Theorem 1.2)**.** *Let $\mathcal{C}$ be the 2-CY category associated to a quiver of finite (ADE) type. Then the stability manifold of $\mathcal{C}$ is connected. Furthermore, up to rotation, every stability condition is in the braid group orbit of a standard stability condition.*

Proof. Let τ be an arbitrary stability condition on $\mathcal{C}$. Let $Z\colon K(\mathcal{C}) \to \mathbf{C}$ be its central charge. Perturb τ so that Z maps distinct roots to complex numbers of distinct arguments. Note that the perturbed τ lies in the same connected component of the stability manifold as the original τ. There is now a unique spherical τ-stable object of every phase.

We have shown in Proposition 5.1 that every spherical object is in the braid group orbit of the simple objects of the standard heart. Therefore the subgroup G_τ of $\operatorname{Aut}(\mathcal{C})$ generated by twists in τ-stable spherical objects is a subgroup of the image of the braid group in $\operatorname{Aut}(\mathcal{C})$.

Let $\Phi_\tau \subset \mathbf{R}$ be the set of phases of spherical τ-stable objects. Since there are finitely many roots, Φ_τ consists of integer translates of a finite set, and hence is discrete.

Let Y be the direct sum of the simple objects in the standard heart of $\mathcal{C}$. Note that Y satisfies the hypotheses of Proposition 5.4. By Proposition 5.4, there is a stability condition ω in the braid group orbit of τ such that Y is in the $[\alpha, \alpha + 1)$ heart of ω. Let ω' be the rotation of ω by α, so that Y lies in the $[0, 1)$ heart of ω'. Note that ω' is in the same connected component as ω.

The direct summands of Y generate the standard heart of $\mathcal{C}$ (under taking extensions). Therefore, the $[0,1)$ heart of ω' contains the standard heart. Since both hearts are hearts of a t-structure, they must in fact be equal. In other words, ω' is a standard stability condition.

Let $\operatorname{Stab}_0 \mathcal{C}$ be the connected component of $\operatorname{Stab} \mathcal{C}$ that contains the standard stability conditions. We have shown that an arbitrary $\tau \in \operatorname{Stab} \mathcal{C}$ is in the braid group orbit of a stability condition ω in $\operatorname{Stab}_0 \mathcal{C}$. But we know by [12, § 4] that the braid group preserves the connected component $\operatorname{Stab}_0 \mathcal{C}$. Furthermore, every $\tau \in \operatorname{Stab}_0$ is, up to rotation, in the braid group orbit of a standard stability condition . Hence, we conclude that $\operatorname{Stab} \mathcal{C} = \operatorname{Stab}_0 \mathcal{C}$ and that every stability condition is, up to rotation, in the braid group orbit of a standard one. □

References

[1] M. Abouzaid and I. Smith. Exact Lagrangians in plumbings. *Geom. Funct. Anal.*, 22(4):785–831, 2012.

[2] T. Adachi, Y. Mizuno, and D. Yang. Discreteness of silting objects and t-structures in triangulated categories. *Proc. Lond. Math. Soc. (3)*, 118(1):1–42, 2019.

[3] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Grad. Texts Math.* New York, NY: Springer, 2005.

[4] A. I. Bondal and M. M. Kapranov. Enhanced triangulated categories. *Math. USSR, Sb.*, 70(1):93–107, 1991.

[5] T. Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007.

[6] T. Bridgeland. Stability conditions and Kleinian singularities. *Int. Math. Res. Not.*, 2009(21):4142–4157, 2009.

[7] T. Bridgeland, A. King, and M. Reid. The McKay correspondence as an equivalence of derived categories. *J. Am. Math. Soc.*, 14(3):535–554, 2001.

[8] T. Etgü and Y. Lekili. Koszul duality patterns in Floer theory. *Geom. Topol.*, 21(6):3313–3389, 2017.

[9] R. S. Huerfano and M. Khovanov. A category for the adjoint representation. *J. Algebra*, 246(2):514–542, 2001.

[10] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 1992.

[11] D. Huybrechts. Stability conditions via spherical objects. *Math. Z.*, 271(3-4):1253–1270, 2012.

[12] A. Ikeda. Stability conditions for preprojective algebras and root systems of Kac-Moody Lie algebras, 2014. `https://arxiv.org/abs/1402.1392v1`.

[13] A. Ishii, K. Ueda, and H. Uehara. Stability conditions on A_n-singularities. *J. Differential Geom.*, 84(1):87–126, 2010.

[14] A. Ishii and H. Uehara. Autoequivalences of derived categories on the minimal resolutions of A_n-singularities on surfaces. *J. Differential Geom.*, 71(3):385–435, 2005.

[15] B. Keller. Deriving DG categories. *Ann. Sci. Éc. Norm. Supér. (4)*, 27(1):63–102, 1994.

[16] B. Keller. On differential graded categories. In *Proceedings of the international congress of mathematicians (ICM), Madrid, Spain, August 22–30, 2006. Volume II: Invited lectures*, pages 151–190. Zürich: European Mathematical Society (EMS), 2006.

[17] A. M. Licata and H. Queffelec. Braid groups of type ADE, garside monoids, and the categorified root lattice, Mar. 2017. Preprint.

[18] V. Mazorchuk, S. Ovsienko, and C. Stroppel. Quadratic duals, Koszul dual functors, and applications. *Trans. Amer. Math. Soc.*, 361(3):1129–1172, 2009.

[19] P. Seidel and R. Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108(1):37–108, 2001.

[20] I. Smith and M. Wemyss. Double bubble plumbings and two-curve flops. 2020.

[21] The Stacks Project Authors. Stacks Project. `http://stacks.math.columbia.edu`, 2022.

[22] R. P. Thomas. Stability conditions and the braid group. *Comm. Anal. Geom.*, 14(1):135–161, 2006.

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