

SOME REMARKS ABOUT THE FAITHFULNESS OF THE BURAU REPRESENTATION OF ARTIN–TITS GROUPS

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ABSTRACT. We discuss the extension of the faithfulness question for the Burau representation of braid groups to the case of Artin–Tits groups. We prove that the Burau representation is not faithful in affine type A_3 , and not faithful over several finite rings in type D_4 , using an algorithmic approach based on categorical methods that generalize Bigelow’s curve strategy outside of type A .

1. INTRODUCTION

The Burau representation, that was defined 90 years ago [Bur35], has for long been the major tool in the study of braids, in relation to the Alexander polynomials of knots. Despite meritorious efforts, its faithfulness is not completely known: it is a classical result (see [MP69] and the discussion in [Bir74]) that it is faithful in type A_2 (3-strand braid group), but the fact that it is not faithful in type A_4 and above was only discovered step by step in the 90’s [Moo91, LP93, Big99]. And type A_3 still resists all efforts [CL97, BT18, Dat22, GWY23], including computer-based ones [FS19] (see also the discussion in [BB21]).

In this paper we build on the observation that the definition of the Burau representation makes perfect sense outside of type A . In this larger generality, faithfulness is open in a fairly large class of cases. We investigated the first cases using computational methods, generalizing Bigelow’s curve-based approach outside of type A , where Khovanov–Seidel’s category [KS02] replaces curves when we lack geometric foundations. It should be highlighted that despite its theoretical definition, this categorical representation is completely computable.

In Theorem 4.1, we exhibit a counterexample to faithfulness in affine type A_3 . In Theorem 6.4, we exhibit a range of counterexamples over several finite rings in type D_4 . These counterexamples were found by running Python and Sage searches on the Australian National Computational Infrastructure (NCI Australia) [Aus19]. The counterexample in affine type A_3 was found using a variation of Bigelow’s strategy from [Big99]. The counterexamples in type D_4 were found by mixing variants of Bigelow’s strategy with the one developed by Gibson, Williamson and Yacobi in [GWY23]. Our code is available on a Git repository [BQ24]. To our surprise, all other cases resisted our investigations over $\mathbb{Z}$.

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2. THE (GENERALISED) BURAU REPRESENTATION

In this section we recall the definition of the Burau representation for the Artin–Tits group associated to any Coxeter diagram. Recall that such a Coxeter diagram Γ is a loop-free finite graph with vertex set $V(\Gamma)$ and unoriented edges. Further, each edge $i - j$ is labelled by $m_{i,j}$ in $\{3, \dots, \infty\}$. Then the associated Artin–Tits group $\mathbf{B}(\Gamma)$ is the group presented as:

$$\mathbf{B}(\Gamma) := \langle \sigma_i \text{ for } i \in V(\Gamma) \mid \underbrace{\sigma_i \sigma_j \cdots}_{m_{i,j}} = \underbrace{\sigma_j \sigma_i \cdots}_{m_{i,j}} \rangle.$$

By convention, $m_{i,j} = \infty$ means that there is no relation involving the letters σ_i and σ_j . If i and j are not adjacent, we set $m_{i,j} = 2$.

The associated Coxeter group has the same presentation, with the added relation that all generators are of order 2:

$$G_\Gamma := \mathbf{B}(\Gamma) / \langle \sigma_i^2 = 1 \rangle.$$

We recall the *geometric representation* of the Coxeter group G_Γ (see, e.g., [BB05, Sect. 4.2]), from which one can study the group properties. The underlying complex vector space $V_{1,\Gamma}$ is freely generated by the roots $\{\alpha_i \mid i \in V(\Gamma)\}$:

$$V_{1,\Gamma} = \mathbb{C}\langle \alpha_i \rangle_{i \in V(\Gamma)},$$

and is equipped with a non-degenerate pairing as follows:

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j; \\ 0 & \text{if } m_{ij} = 2 \text{ (} i \text{ and } j \text{ not adjacent);} \\ -2 \cos(\frac{\pi}{m_{ij}}) & \text{if } m_{ij} \neq \infty; \\ -2 & \text{if } m_{ij} = \infty. \end{cases}$$

Note that the $m_{ij} = 2$ case is compatible with the definition for general m_{ij} . Now G_Γ acts on $V_{1,\Gamma}$ by reflections via the following formula:

$$s_i(\alpha_j) = \alpha_j - \langle \alpha_i, \alpha_j \rangle \alpha_i.$$

A key point is that this representation is faithful (see for example [BB05, Thm 4.2.7]).

Passing to Artin–Tits groups, this geometric representation q -deforms into the Burau representation. Let us first consider $V_{q,\Gamma}$, which is the $\mathbb{C}(q)$ -vector space $\mathbb{C}(q) \otimes_{\mathbb{C}} V_{1,\Gamma}$. Then $V_{q,\Gamma}$ has a q -deformed pairing, defined as follows. On the basis elements, we set:

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 1 + q^2 & \text{if } i = j \\ 2q \cos(\frac{\pi}{m_{ij}}) & \text{if } i \neq j \text{ and } m_{ij} < \infty \\ 2q & \text{if } m_{ij} = \infty. \end{cases}$$

The pairing is extended to $V_{q,\Gamma}$ by $\mathbb{C}$ -bilinearity, and sesquilinearity with respect to q as follows:

$$\langle qx, y \rangle = q^{-1} \langle x, y \rangle \text{ and } \langle x, qy \rangle = q \langle x, y \rangle$$

Taking $q = -1$ takes us back to the undeformed setting.

Definition 2.1. *The Burau representation is an action of $\mathbf{B}(\Gamma)$ on $V_{q,\Gamma}$ defined on generators $\sigma_i \in \mathbf{B}(\Gamma)$ by:*

$$\sigma_i(\alpha_j) = \alpha_j - \langle \alpha_i, \alpha_j \rangle \alpha_i.$$

One can check the following formula for the inverses of the generators:

$$\sigma_i^{-1}(\alpha_j) = \alpha_j - q^{-2} \langle \alpha_j, \alpha_i \rangle \alpha_i.$$

Example 2.2. *In simply-laced type, one can check that:*

$$\sigma_i(\alpha_j) = \begin{cases} -q^2 \alpha_i & \text{if } i = j, \\ \alpha_i - q \alpha_j & \text{if } m_{i,j} = 3, \\ \alpha_j & \text{otherwise.} \end{cases}$$

3. SUMMARY OF RESULTS AND APPROACHES

3.1. Faithfulness questions for generalized Burau representations. Studying the faithfulness of the Burau representation in type A has for long been a tantalizing question. It is known to be faithful in type A_2 , which corresponds to the 3-strand braid group (see [MP69] or [Bir74] and references therein), and unfaithful for at least 5 strands [Moo91, LP93, Big99]. The 4-strand case (corresponding to type A_3) is open.

In light of the generalization of the Burau representation to larger types, it is a natural question to ask about faithfulness in other types. With type A_3 in mind, being able to identify a kernel in nearby types (like $D_4, \widetilde{A}_3, \dots$) might help identifying properties of elements in the kernel that would yield proof strategies for faithfulness.

In Theorem 4.1, we show that the Burau representation is not faithful over $\mathbb{Z}$ in affine type $\widetilde{A}_3$. In Theorem 6.4 we show that the Burau representation is not faithful in type D_4 over finite rings $\mathbb{Z}/p\mathbb{Z}$ with $p \leq 16$.

Towards approaching more general types, the following observation is clear from the fact that Artin–Tits groups include into each other in a way that is compatible with the Burau representation. Recall

that a subgraph of an undirected graph is called a *full subgraph* if its vertex set is a subset of the vertices, and its edge set contains all of the edges in the original graph between any pair of vertices in the subset.

Lemma 3.1. *If a Coxeter graph Γ contains A_4 or $\widetilde{A}_3$ as a full subgraph, then the Burau representation of B_Γ is not faithful.*

Proof. We know from existing results that the Burau representation is not faithful in type A_n for $n = 4$ and beyond, and we know from Theorem 4.1 that it is not faithful in type $\widetilde{A}_3$. Thus for any graph that contains these as full subgraphs, the kernel of the Burau representation will be non-trivial. $\square$

Combined with the fact that faithfulness is not known in type A_3 , this lemma leaves us with a large range of open cases. Namely, all Coxeter graphs Γ that do not contain A_4 or $\widetilde{A}_3$ as full subgraphs. Among these, we investigated the following ones (we restricted to the simply-laced case).



A number of other graphs are possible, including non-simply-laced ones, as well as more complicated simply-laced ones such as many-pointed stars (continuing from D_4 and $\widetilde{D}_4$), and complete graphs K_n of higher order. As far as we are aware, all of these cases (and other similar ones) remain open over $\mathbb{Z}$. It was surprising to us that none of the other possibilities gave us any answers.

3.2. Proof strategies (after Bigelow). For both Theorems 4.1 and 6.4, we used adaptations of Bigelow's proof strategy from [Big99] for the lack of faithfulness in type A_4 and above. We recall this strategy now, together with our modified versions. The proof is based on the fact that the braid group $\mathbf{B}_n = \mathbf{B}_{A_{n-1}}$ is the mapping class group of the disk with n points removed, which acts transitively on simple curves connecting two of the points. See Figure 1 for an illustration; note that the numbering of the points is fixed, and we do not carry it along when we act on curves. The centralizer of the full set of curves is the center of the braid group, generated by the full twist. Furthermore, to each curve γ we

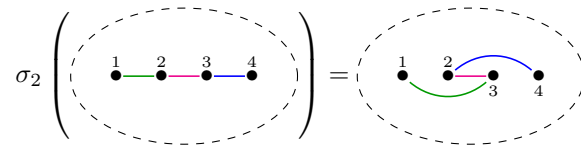


FIGURE 1. The action of the braid element $\sigma_2 = \tau_{(2,3)}$ on the basic curves

can assign the braid τ_γ , which is the anticlockwise half Dehn twist along the curve.

Let V_q denote the Burau representation (see Theorem 2.1) of the chosen Coxeter type (A_{n-1} in this case). There is a way (which we do not detail here, but is explained, e.g. in [KS02] and in [Big99]) to assign an element of V_q to any curve. The basic curves $(i, i+1)$ are sent to α_i . For more complicated curves, the coefficient of α_i roughly counts how many times the curve navigates between points i and $i+1$, by computing minimal intersection numbers with certain basic curves. Furthermore, each such intersection number is attached to a power of q based on the total winding number of the curve around the outermost disk from the starting point up to the current point. Thus the coefficient of each α_i is a certain graded intersection number. This process is compatible with the pairing, in the following sense: it sends a graded version of the number of intersections between curves to the pairing.

Bigelow's criterion to disprove faithfulness [Big99, Theorem 1.4] then expresses as follows: the representation is faithful if and only if there does not exist a pair of curves γ_1 and γ_2 that:

- do intersect non-trivially;
- but have trivial pairing in V_q : $\langle \gamma_1, \gamma_2 \rangle = 0$.

Indeed if such a pair can be found, then the commutator of the corresponding half Dehn twists $[\tau_{\gamma_1}, \tau_{\gamma_2}]$, is a non-trivial element in the kernel. Reciprocally, if β is in the kernel, then a pair of the kind $((i, i+1), \beta((i+2, i+3)))$ will meet the above criteria.

Remark 3.2. *The curve-based strategy does not immediately extend outside of type A, where one lacks geometric models. The key idea behind this paper is that the Khovanov–Seidel categorical representation [KS02] provides a natural replacement for curves, namely spherical objects in a certain category. This categorical representation is computable via the code in [BD24], and thus we are able to run computer searches for counterexamples.*

We will develop this idea and its consequences in Sections 4 and 5. In particular, we state two versions of Bigelow's criterion for curves to generate a non-trivial element of the kernel (Theorem 5.1 and Theorem 5.2), where the non-triviality uses information about the space of morphisms at the categorical level.

Finally, in Section 6, we find another condition for a curve-like object to generate a non-trivial element of the kernel (see Theorem 6.2). We mix this with Gibson, Williamson and Yacobi's bucket search [GWY23] to explore faithfulness in type D_4 .

The three conditions provide variations of Bigelow's criterion, and extend it in the situations where geometric arguments are not readily available. Here are the three basic ideas we use, expressed in the classical curve context: for curves γ_1 and γ_2 and their associated half Dehn twists τ_{γ_1} and τ_{γ_2} ,

- in Theorem 5.1: if $\langle \gamma_1, \gamma_2 \rangle = 0$, then $[\tau_{\gamma_1}, \tau_{\gamma_2}]$ is in the kernel of the Burau representation;
- in Theorem 5.2: if $\langle \gamma_1, \gamma_2 \rangle = \pm q^r$, then $\tau_{\gamma_1} \tau_{\gamma_2} \tau_{\gamma_1} \tau_{\gamma_2}^{-1} \tau_{\gamma_1}^{-1} \tau_{\gamma_2}^{-1}$ is in the kernel of the Burau representation;
- in Theorem 6.3: if γ_1 and γ_2 are equal in V_q , then $\tau_{\gamma_1} \tau_{\gamma_2}^{-1}$ is in the kernel of the Burau representation.

Outside of type A, and in particular in the case of highly connected graphs, Theorem 5.2 might prove more useful. For example in the case of the complete graphs, no two simple roots pair together to zero, which makes the original criterion harder to use.

4. THE AFFINE A_3 CASE

Here is our first main result.

Theorem 4.1. *The Burau representation is not faithful in type $\widetilde{A}_3$.*

Proof. Consider the following braids:

$$\begin{aligned} a &= \sigma_3^2 \sigma_4 \sigma_3 \sigma_2 \sigma_1 \sigma_3^{-1} \sigma_4 \sigma_3 \sigma_2 \sigma_1^{-2} \sigma_4, \\ b &= \sigma_1^2 \sigma_2^{-1} \sigma_4 \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_4^{-1} \sigma_3 \sigma_1 \sigma_4 \sigma_1 \sigma_2^{-1} \sigma_4^{-2} \sigma_3, \\ \alpha &= a \sigma_3 a^{-1}, \\ \beta &= b \sigma_2 b^{-1}. \end{aligned}$$

Then we claim that $[\alpha, \beta]$ is a non-trivial element in the kernel of the Burau representation.

Using Bigelow's trick, one can prove that $[\alpha, \beta]$ is indeed in the kernel by checking that the pairing between ae_3 and be_2 is zero, where e_2 and e_3 are the second and third basis vector. Non-triviality follows from the fact that the Hom space between the two corresponding spherical objects aP_3 and bP_2 is non-trivial. $\square$

Remark 4.2. *Our original computer calculations output the following braid words for a and b respectively:*

$$\begin{aligned} a' &= \sigma_3 \sigma_1 \sigma_2 \sigma_1 \sigma_3^{-1} \sigma_4 \sigma_2 \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_1^{-2} \sigma_4 \\ b' &= \sigma_1 \sigma_4^{-1} \sigma_1^2 \sigma_3^{-2} \sigma_2^{-1} \sigma_4 \sigma_1 \sigma_3^{-1} \sigma_2 \sigma_4^{-1} \sigma_3 \sigma_1 \sigma_4 \sigma_1 \sigma_2^{-1} \sigma_4^{-2} \sigma_3. \end{aligned}$$

We can transform the pair (a', b') into the pair (a, b) used in Theorem 4.1 by braid moves, simultaneous letter cancellation, as well as by moving some letters from the front of b' to the front of a' . Explicitly, we follow the following steps.

- (1) Reduce the sub-expression $\sigma_2 \sigma_3 \sigma_2 \sigma_3^{-1}$ appearing in a' to $\sigma_3 \sigma_2$.

- (2) Swap $\sigma_3\sigma_1$ at the start of a' to $\sigma_1\sigma_3$, and then cancel σ_1 from the front of both words.
- (3) Swap $\sigma_1^2\sigma_3^{-2}$ to $\sigma_3^{-2}\sigma_1^2$ in b' , and then move the expression $\sigma_4^{-1}\sigma_3^{-2}$ appearing in the front of b' to the front of a as the inverse expression $\sigma_3^2\sigma_4$.

We make this transformation because the pair (a, b) is much simpler, and produces a cleaner drawing as illustrated below.

Although we used the Khovanov–Seidel categorical representation to find this counterexample, the case of affine type A does, in fact, have a geometric model (see for instance [GTW15], especially Sections 4 and 5). The braid group of affine type A_n is a subgroup of the mapping class group of an annulus with $n + 1$ punctures. We consider the action of this braid group on curves that begin and end at two punctures, do not self-intersect, and do not intersect any of the punctures in their interior. Thus our objects aP_3 and bP_2 can be represented by such curves as well.

It is convenient to draw the curves on the universal cover of the annulus with $n + 1 = 4$ punctures. The universal cover is then an infinite punctured strip, where each puncture lifts to $\mathbb{Z}$ -many copies of itself. A curve on the annulus from point p to q can be lifted uniquely to a curve starting at any lift of p ; it will then end at some lift of q .

Label the four punctures in the annulus successively as 1, 2, 3, 4. Then P_1 , P_2 , and P_3 are represented by the shortest curves joining $1 \leftrightarrow 2$, $2 \leftrightarrow 3$, and $3 \leftrightarrow 4$ respectively, while P_4 is the curve joining $4 \leftrightarrow 1$ going around the hole of the annulus, as shown in Figure 2.

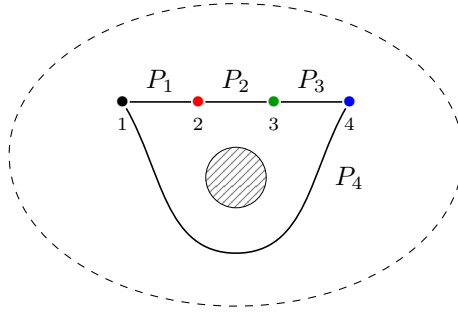


FIGURE 2. The basic curves P_1 , P_2 , P_3 , and P_4 as drawn on the annulus.

In the universal cover, the curve P_i is then represented by the straight line joining any lift of the point i to the closest lift of the point $i + 1$ (modulo 4). The Dehn twist in a curve on the annulus is performed in the universal cover simultaneously around every lift of the curve. With this correspondence, our curves aP_3 and bP_2 are drawn on the universal cover as shown in Figures 3 and 4. It is evident from the pictures that the curves intersect: this is easiest to verify close to the left hand endpoint of the top curve, and also the right hand endpoint of the bottom curve.

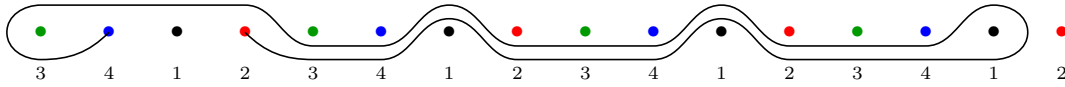


FIGURE 3. The curve aP_3 drawn in the universal cover of the annulus with four marked points.

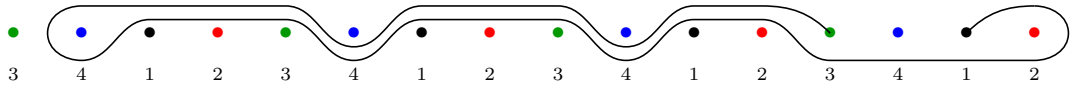


FIGURE 4. The curve bP_2 drawn in the universal cover of the annulus with four marked points.

Remark 4.3. It is a classical result that Artin–Tits groups in type $\widetilde{A}_n$ embed in braid groups of type A_{n+1} : the former can be represented by braids in an annulus, the core of which can be made into a fixed strand of a classical braid (see for example [All02]). First, this embedding suggests a curve-based approach to the above result. We did not use such an approach computationally, since we wanted to investigate more cases than those accessible by curve-based methods. Second, this embedding raises the question of whether the lack of faithfulness claimed above is a consequence of the lack of faithfulness in type A_4 . There do not seem to be direct arguments in favor of this however, as the Burau pairing in affine type A is not the one inherited by the inclusion of groups.

5. STRATEGY AND COMPUTATIONS OVER $\mathbb{Z}$

In this section we expand on Theorem 3.2 to explain the design of the search that led to Theorem 4.1. Our strategy is based on the one used by Bigelow, with curves replaced by spherical objects in Khovanov–Seidel’s category. It should be noted that Khovanov–Seidel’s representation can be easily defined in any simply-laced type, but that no faithfulness result is known outside of types A , D , E and $\tilde{A}$.

We recall some important features of the Khovanov–Seidel categorical representation, referring the reader to [KS02] or the survey [Que24] for further details. Fix a Coxeter diagram Γ . Let A_Γ be the zigzag algebra of Γ (see, e.g., [HK01, Section 5] as well as [BDL20, Remark 6.6]). Let $\mathcal{C}_\Gamma$ be the bounded homotopy category of finitely-generated graded projective modules over A_Γ . We set $\{P_i \mid i \in V(\Gamma)\}$ to be the projective indecomposable A_Γ -modules. Following [KS02], we can define an action of the Artin–Tits group $\mathbf{B}(\Gamma)$ on $\mathcal{C}_\Gamma$. The resulting action of $\mathbf{B}(\Gamma)$ on the Grothendieck group $K_0(\mathcal{C}_\Gamma)$ recovers the Burau representation $V_q = V_{q,\Gamma}$.

The morphisms in $\mathcal{C}_\Gamma$ are bigraded by internal graded degree and homological degree. We write $\text{Hom}_{\mathcal{C}_\Gamma}^{g,h}(X, Y)$ to mean the morphisms from X to Y of graded degree g and homological degree h . That is,

$$\text{Hom}_{\mathcal{C}_\Gamma}^{g,h}(X, Y) = \text{Hom}_{\mathcal{C}_\Gamma}(X, Y\{g\}[h]),$$

where the functors $\{g\}, [h]: \mathcal{C}_\Gamma \rightarrow \mathcal{C}_\Gamma$ denote the grading shift and homological shift by g and h respectively. We write $\text{Hom}_{\mathcal{C}_\Gamma}^{\bullet,\bullet}(X, Y)$ to mean the total bigraded hom space. That is,

$$\text{Hom}_{\mathcal{C}_\Gamma}^{\bullet,\bullet}(X, Y) = \bigoplus_{g,h \in \mathbb{Z}} \text{Hom}_{\mathcal{C}_\Gamma}^{g,h}(X, Y).$$

The spaces of bigraded morphisms enjoy the following duality for any $g, h \in \mathbb{Z}$ and objects X, Y of $\mathcal{C}_\Gamma$:

$$\text{Hom}_{\mathcal{C}_\Gamma}^{g,h}(X, Y) \cong \text{Hom}_{\mathcal{C}_\Gamma}^{2-g, -h}(Y, X)^\vee.$$

For $X, Y \in \mathcal{C}_\Gamma$, the pairing on $K_0(\mathcal{C}_\Gamma) = V_q$ between $[X]$ and $[Y]$ is simply the $(q, -1)$ Euler characteristic of the total hom space. That is:

$$\langle [X], [Y] \rangle = \sum_{g \in \mathbb{Z}} q^g \left(\sum_{h \in \mathbb{Z}} (-1)^h \dim \text{Hom}_{\mathcal{C}_\Gamma}^{g,h}(X, Y) \right).$$

Note that by the duality between the hom-spaces, the pairing $\langle [X], [Y] \rangle$ is related to $\langle [Y], [X] \rangle$. More precisely, suppose that $\langle [X], [Y] \rangle = p(q)$ for some polynomial expression in $\mathbb{Z}[q, q^{-1}]$. Then $\langle [Y], [X] \rangle = q^2 p(q^{-1})$.

A special role is played by the *(2-)spherical objects* in $\mathcal{C}_\Gamma$: these are the objects X such that:

- $\text{End}_{\mathcal{C}_\Gamma}^{\bullet,h}(X) = 0$ for $h \neq 0$; and
- the algebra $\text{End}_{\mathcal{C}_\Gamma}^{\bullet,0}(X)$ is isomorphic to the cohomology algebra of the 2-sphere.

Each spherical object $X \in \mathcal{C}_\Gamma$ gives rise to an autoequivalence $\tau_X: \mathcal{C}_\Gamma \xrightarrow{\sim} \mathcal{C}_\Gamma$, called the *spherical twist* with respect to X . In particular, the generating projective modules $\{P_i \mid i \in V(\Gamma)\}$ are always spherical, and the associated twists τ_{P_i} satisfy the defining relations of the group $\mathbf{B}(\Gamma)$ up to equivalence.

In the situations when we have geometric models such as in type A , the spherical objects (up to shift) are the ones that can be recovered from curves, as explained in Section 3.2. Recall also that the pairing between two spherical objects is realised by the graded intersection number between the corresponding curves. Moreover, the spherical twist in a spherical object corresponds to a half Dehn twist in the corresponding curve. For these reasons, in the general situation, we regard the spherical objects of $\mathcal{C}_\Gamma$ as analogues of curves. We now explain how we adapt Bigelow’s strategy to these more general situations.

We now describe two versions of Bigelow’s criterion, each of which is useful depending on the situation. The first criterion (Theorem 5.1) is very similar to the one used by Bigelow [Big99, Theorem 1.4], while the second one (Theorem 5.2) is new. Furthermore, the proof of the first one is similar to, but easier than the proof of the second one. We thus only prove the second criterion here, and leave the proof of the first criterion to the reader. We emphasize again that we are replacing the geometric condition in Bigelow’s criterion by a homological one, in which curves are replaced by spherical objects.

Lemma 5.1. *Assume we have two spherical objects X_1 and X_2 in $\mathcal{C}_\Gamma$, so that:*

- $\text{Hom}_{\mathcal{C}_\Gamma}^{\bullet,\bullet}(X_1, X_2) \neq \{0\}$; and
- $\langle [X_1], [X_2] \rangle = 0$ in V_q .

Then the commutator $[\tau_{X_1}, \tau_{X_2}] = \tau_{X_1} \tau_{X_2} \tau_{X_1}^{-1} \tau_{X_2}^{-1}$ of their associated spherical twists is a non-trivial element of the kernel of the Burau representation.

Lemma 5.2. *Assume we have two spherical objects X_1 and X_2 , so that:*

- $\dim \text{Hom}_{\mathcal{C}_\Gamma}^{\bullet, \bullet}(X_1, X_2) > 1$; and
- $\langle [X_1], [X_2] \rangle = q$ in V_q .

Then $\tau_{X_1} \tau_{X_2} \tau_{X_1}^{-1} \tau_{X_2}^{-1} \tau_{X_1}^{-1} \tau_{X_2}^{-1}$ is a non-trivial element of the kernel of the Burau representation.

Before we prove this statement, recall from [KS02, Section 2d] that applying the spherical twist in an object X is the functor of tensoring with a certain complex of bimodules over the zigzag algebra A_Γ , namely

$$\tau_X \leftrightarrow (X \otimes X^\vee \rightarrow A_\Gamma),$$

where the map from $X \otimes X^\vee \rightarrow A_\Gamma$ is simply multiplication. Henceforth, let us identify τ_X with the complex $X \otimes X^\vee \rightarrow A_\Gamma$, so that composition of twists turns into tensoring these complexes together over A_Γ . Furthermore, by standard results we have a natural isomorphism of bigraded vector spaces

$$X^\vee \otimes_{A_\Gamma} Y \cong \text{Hom}_{\mathcal{C}_\Gamma}^{\bullet, \bullet}(X, Y).$$

Proof of Theorem 5.2. The fact that the expression $\tau_{X_1} \tau_{X_2} \tau_{X_1}^{-1} \tau_{X_2}^{-1} \tau_{X_1}^{-1} \tau_{X_2}^{-1}$ lies in the kernel of the Burau representation is clear from the pairing. It remains to prove that it is non-trivial, which amounts to proving that:

$$\tau_{X_1} \tau_{X_2} \tau_{X_1} \neq \tau_{X_2} \tau_{X_1} \tau_{X_2}.$$

The spherical twists τ_{X_1} and τ_{X_2} can be expressed as follows:

$$\tau_{X_1} = X_1 \otimes X_1^\vee \rightarrow A_\Gamma$$

$$\tau_{X_2} = X_2 \otimes X_2^\vee \rightarrow A_\Gamma.$$

The composition $\tau_{X_1} \tau_{X_2} \tau_{X_1} = \tau_{X_1} \otimes_{A_\Gamma} \tau_{X_2} \otimes_{A_\Gamma} \tau_{X_1}$ is then the following complex of bimodules.

$$\begin{array}{ccccc} & & X_1 \otimes (X_1^\vee \otimes_{A_\Gamma} X_1) \otimes X_1^\vee & \longrightarrow & X_1 \otimes X_1^\vee \\ & \nearrow & & \searrow & \nearrow \\ X_1 \otimes (X_1^\vee \otimes_{A_\Gamma} X_2) \otimes (X_2^\vee \otimes_{A_\Gamma} X_1) \otimes X_1^\vee & \longrightarrow & X_2 \otimes (X_2^\vee \otimes_{A_\Gamma} X_1) \otimes X_1^\vee & \longrightarrow & X_1 \otimes X_1^\vee \longrightarrow A_\Gamma \\ & \searrow & & \nearrow & \searrow \\ & & X_1 \otimes (X_1^\vee \otimes_{A_\Gamma} X_2) \otimes X_2^\vee & \longrightarrow & X_2 \otimes X_2^\vee \end{array}$$

Recall that $X_1^\vee \otimes X_1 \simeq \mathbb{C} \oplus \mathbb{C}\langle 2 \rangle$ because X is spherical. By the assumption on the morphisms from X_1 to X_2 as well as the pairing $\langle [X_1], [X_2] \rangle$, we have

$$X_1^\vee \otimes_{A_\Gamma} X_2 \cong \mathbb{C}\langle 1 \rangle \oplus W_1$$

for some non-trivial bigraded vector space W_1 that has trivial bigraded dimension (i.e. $(q, -1)$ -dimension as in the definition of the pairing).

Observe by the duality that $\dim \text{Hom}_{\mathcal{C}_\Gamma}^{\bullet, \bullet}(X, Y) > 1$ if and only if $\dim \text{Hom}_{\mathcal{C}_\Gamma}^{\bullet, \bullet}(Y, X) > 1$. Also, $\langle [X_1], [X_2] \rangle = q$ implies that $\langle [X_2], [X_1] \rangle = q$ as well. So we also have that

$$X_2^\vee \otimes_{A_\Gamma} X_1 \simeq \mathbb{C}\langle 1 \rangle \oplus W_2$$

for some non-trivial bigraded vector space W_2 that has trivial bigraded dimension.

The leftmost position now becomes

$$\begin{aligned} X_1 \otimes (\mathbb{C}\langle 1 \rangle \oplus W_1) \otimes (\mathbb{C}\langle 1 \rangle \otimes W_2) \otimes X_1^\vee &= (X_1 \otimes \mathbb{C}\langle 2 \rangle \otimes X_1^\vee) \oplus \\ &\quad (X_1 \otimes (W_1\langle 1 \rangle \oplus W_2\langle 1 \rangle \oplus (W_1 \otimes W_2)) \otimes X_1^\vee). \end{aligned}$$

The top-left position becomes

$$X_1 \otimes (X_1^\vee \otimes_{A_\Gamma} X_1) \otimes X_1^\vee \cong (X_1 \otimes \mathbb{C} \otimes X_1^\vee) \oplus (X_1 \otimes \mathbb{C}\langle 2 \rangle \otimes X_1^\vee).$$

We can now apply Gaussian elimination of complexes (see, e.g. [BN07, Lemma 4.2]). The term $X_1 \otimes \mathbb{C}\langle 2 \rangle \otimes X_1^\vee$ in the leftmost position cancels with the same term in the top-left position, while the term $X_1 \otimes \mathbb{C} \otimes X_1^\vee = X_1 \otimes X_1^\vee$ in the top-left position cancels with the top-right position. After these cancellations, the top row disappears, and we obtain the following reduced complex.

$$\begin{array}{ccccc} X_1 \otimes (W_1\langle 1 \rangle \oplus W_2\langle 1 \rangle \oplus (W_1 \otimes W_2)) \otimes X_1^\vee & \longrightarrow & X_2 \otimes (\mathbb{C}\langle 1 \rangle \oplus W_2) \otimes X_1^\vee & \longrightarrow & X_1 \otimes X_1^\vee \longrightarrow A_\Gamma \\ & \searrow & & \nearrow & \searrow \\ & & X_1 \otimes (\mathbb{C}\langle 1 \rangle \oplus W_1) \otimes X_2^\vee & \longrightarrow & X_2 \otimes X_2^\vee \end{array}$$

In the new complex, all of the terms except for the leftmost one are symmetric under exchanging 1 and 2. Setting $U = W_1\langle 1 \rangle \oplus W_2\langle 1 \rangle \oplus (W_1 \otimes W_2)$, we see that the left-most terms in $\tau_{X_1}\tau_{X_2}\tau_{X_1}$ and $\tau_{X_2}\tau_{X_1}\tau_{X_2}$ are, respectively, $X_1 \otimes U \otimes X_1^\vee$ and $X_2 \otimes U \otimes X_2^\vee$. Since U is non-trivial and X_1 and X_2 are not isomorphic, these two terms are different. We see that

$$\tau_{X_1}\tau_{X_2}\tau_{X_1} \neq \tau_{X_2}\tau_{X_1}\tau_{X_2}.$$

Hence the desired expression $\tau_{X_1}\tau_{X_2}\tau_{X_1}\tau_{X_2}^{-1}\tau_{X_1}^{-1}\tau_{X_2}^{-1}$ is non-trivial, because it is sent to a non-trivial element under the Khovanov–Seidel representation. $\square$

The second version of the criterion (Theorem 5.2) is especially useful when the graph is highly connected, when there might not be many pairs of braids that commute. It is interesting to notice that these two criteria combined will detect the fact that the kernel of the Burau representation is bigger than the kernel of the Khovanov–Seidel representation. When the latter is faithful (in particular in spherical type and affine type A), this gives an effective criterion to detect the kernel of the Burau representation. This follows from the following lemma.

Lemma 5.3. *Assume that a braid β acts on $\mathcal{C}_\Gamma$ non-trivially, but is in the kernel of the Burau representation. Then there exist $(i, j) \in V(\Gamma)$ such that one of the following holds:*

- $\langle \alpha_i, \alpha_j \rangle = 0$ and $\dim(\text{Hom}_{\mathcal{C}_\Gamma}^\bullet(\beta P_i, P_j)) > 0$, in which case Theorem 5.1 applies; or
- $\langle \alpha_i, \alpha_j \rangle = q$ and $\dim(\text{Hom}_{\mathcal{C}_\Gamma}^\bullet(\beta P_i, P_j)) > 1$, in which case Theorem 5.2 applies.

Here by a non-trivial action, we mean that β does not act as the composition of homological shifts on each block corresponding to each connected component of Γ .

Proof. Since β acts non-trivially on $\mathcal{C}$, in particular there exists $i \in V(\Gamma)$ so that $\beta P_i \neq P_i$. But since β is in the kernel of the Burau representation, $\beta(\alpha_i) = \alpha_i$, which is equivalent to writing the following in $K_0(\mathcal{C})$:

$$\beta([P_i]) = [P_i] \text{ for every } i \in V(\Gamma).$$

Now, filter the complex βP_i by the path length grading, and consider the bottom piece (that comes with highest shift b in grading). One sees a direct sum $\bigoplus_k P_{l_k}\{b\}[h_k]$, and βP_i looks like:

$$\beta P_i = \text{Cone} \left(\bigoplus_k P_{l_k}[h_k] \rightarrow X' \right).$$

with X' made of P_k 's shifted by strictly less than b in path length grading.

If for one of the l_k 's one has $\langle \alpha_i, \alpha_{l_k} \rangle = 0$, then the identity map from P_{l_k} in X to P_{l_k} will generate a non-zero element of $\text{Hom}_{\mathcal{C}_\Gamma}(X, P_{l_k})$. Thus the dimension of this $\text{Hom}_{\mathcal{C}_\Gamma}$ -space is strictly positive and we are done.

If for one of the l_k 's one has $\langle \alpha_i, \alpha_{l_k} \rangle = q$, then again the identity map from P_{l_k} in X to P_{l_k} will generate a non-zero element of $\text{Hom}(X, P_{l_k})$. If P_{l_k} appears more than once (that is, $\exists k'$ with $l_k = l_{k'}$), then the dimension is strictly greater than 2 and we are done. Otherwise, this means that in $[\beta P_i]$, $q^b \alpha_{l_k}$ appears with coefficient $(-1)^{h_k} \neq 0$, which contradicts the fact that $[\beta P_i] = [P_i]$.

This leaves us with the situation where all $l_k = i$. In this case, we do the same process with the top slice, of path length degree $t < b$ (this time, surviving maps will be loop maps). The only case where we cannot conclude is when only P_i 's appear as well.

If βP_i is not concentrated in a single q -degree, that is if $t \neq b$, then any identity map from a P_i in degree t to a P_i in degree b produces a non-zero map in homotopy. One at least of the top and bottom slices must contain two P_i 's in different homological grading: if not, both the top and bottom slices would produce non-zero terms in the Grothendieck group, which would contradict $[\beta P_i] = [P_i]$. This means that there is a non-zero map in odd homological degree in $\text{End}_{\mathcal{C}_\Gamma}^{\bullet, \bullet}(\beta P_i)$, which contradicts the fact that $\text{End}_{\mathcal{C}_\Gamma}^{\bullet, \bullet}(\beta P_i) \simeq \text{End}_{\mathcal{C}_\Gamma}^{\bullet, \bullet}(P_i) \simeq \mathbb{C} \oplus \mathbb{C}\langle 2 \rangle$.

So the only case left is when $b = t$. Then since the complex is indecomposable, one must have $\beta P_i = P_i\{b\}[h]$, and since $[\beta P_i] = [P_i]$, this determines $b = 0$ and $h \in 2\mathbb{Z}$.

The only case where we cannot conclude yet is when $\beta P_j = P_j[h_j]$ for each j , with $h_j \in 2\mathbb{Z}$. If i and j are adjacent in Γ , the fact that $\text{Hom}_{\mathcal{C}_\Gamma}(P_i, P_j) \simeq \text{Hom}_{\mathcal{C}_\Gamma}(\beta P_i, \beta P_j)$ lies in homological degree zero implies that $h_i = h_j$. By connectedness, this implies that $h_i = h_j$ if i and j belong to the same connected component of Γ . But then this means that β acts trivially, so this case does not happen. $\square$

Our strategy of search goes as follows.

- (1) We produce a large set (size approximately 2 million) of q -polynomials for generalized curves, simply by acting on elementary curves with generators of the braid group. We usually classify these curves by their associated root (namely the value of the q -polynomial at $q = 1$, possibly modulo 2 to get a finite list) and a length criterion, for instance the sum of absolute values of coefficients of the q -polynomial.
- (2) We then record the pairs of q -polynomials (γ_1, γ_2) that have trivial pairing (for one or the other version of the criterion). We usually restrict the search to specific pairs of roots to limit the size of the calculation, and also try to avoid duplicates. For example, the pair $(\beta\gamma_1, \beta\gamma_2)$ would be redundant for any braid β , and we try to rule out such cases.
- (3) Finally, we compute the objects in the category that correspond to γ_1 and γ_2 and compute the $\text{Hom}_{\mathcal{C}_r}^\bullet$ -space. If this is non-trivial (depending on which version of the criterion we are using), then we have a counterexample.

Our code is available in the Git repository [BQ24].

6. RANDOM SEARCHES OVER $\mathbb{Z}/p\mathbb{Z}$

In finite type (so, in our case, A_3 and D_4), we adapt the strategy of Gibson, Williamson and Yacobi from [GWY23] to a curve-search random algorithm. Recall that these authors found an explicit braid in the kernel of the Burau representation over $F_5 = \mathbb{Z}/5\mathbb{Z}$ by studying the projective length (or **projlen**) of braids, which is the difference between the top and bottom q -degree of the terms of a matrix. We translate this search in terms of curves and move it from the classical Garside structure to the dual one.

Recall that braid groups of finite type admit a so-called *dual* set of generators, due to Birman-Ko-Lee [BKL98] and Bessis [Bes03]. Let n be the rank of the chosen type. Fix a Coxeter element γ , which is the product of the generators σ_i in some fixed chosen order. In practice, we usually take the product in the order given by the index; for instance, $\gamma = \sigma_1\sigma_2\sigma_3\sigma_4$ for D_4 . Then consider the following subset of $B(\Gamma)$:

$$\mathbb{T} = \{\beta\sigma_i\beta^{-1}, \beta \in \mathbf{B}(\Gamma)\}.$$

Definition 6.1. Define $[1, \gamma]$ to be the set of braids:

$$[1, \gamma] := \{\tau_1 \cdots \tau_k \mid \text{there exist } \tau_{k+1}, \dots, \tau_n \text{ in } \mathbb{T} \text{ such that } \tau_1 \cdots \tau_n = \gamma\}$$

In other words, we consider prefixes of γ written as a minimal length product of lifts of reflections. It follows from Bessis' work that this definition is meaningful. Atoms in this set are $\mathbb{T} \cap [1, \gamma]$.

For $\beta_1, \beta_2 \in [1, \gamma]$, we will say that β_1 *divides* β_2 , written as $\beta_1 \mid \beta_2$, if:

there exists $\beta_3 \in [1, \gamma]$ such that $\beta_2 = \beta_1\beta_3$, or equivalently, if:

there exists $\beta_4 \in [1, \gamma]$ such that $\beta_2 = \beta_4\beta_1$.

That is, left and right divisibility are the same.

Associated to the set of generators $[1, \gamma]$ is a normal form (Garside normal form), which in our case allows us to generate non-trivial braids of increasing complexity. We will first tweak this normal form for it to be adapted to the curve search. Before doing so, let us slightly change the Burau representation to an equivalent form that is more suited to these new generators. Recall that our choice of Coxeter element γ corresponds to a fixed order on the Artin generators; in this case, by increasing index. Now, we restrict to finite type, and define the following:

$$\langle \alpha_i, \alpha_j \rangle_d = \begin{cases} 1 + q & \text{if } i = j \\ 1 & \text{if } i \leq j \text{ and } m_{ij} = 3 \\ q & \text{if } j \leq i \text{ and } m_{ij} = 3. \end{cases}$$

The explicit action of generator thus becomes, for $\alpha \in V_q$:

$$(1) \quad \begin{aligned} \sigma_i(\alpha) &= \alpha - \langle \alpha_i, \alpha \rangle_d \alpha_i \\ \sigma_i^{-1}(\alpha) &= \alpha - q^{-1} \langle \alpha, \alpha_i \rangle_d \alpha_i. \end{aligned}$$

We define the *spread* of a matrix as the difference between the top q -degree of all its entries, and the bottom q -degree of all of its entries. A nice feature of this new form is that images of elements of $[1, \gamma]$ only have degree 0 and 1, and thus spread 0 (only the id and γ) or 1.

It is proven in [LQ21] that the categorical spread controls the Garside length, which is equivalent to the categorical faithfulness of the Burau representation. In this case, the goal is to find braids with non-zero Garside length but spread 0. We can obtain all braids from braids in the positive monoid by multiplication by a power of γ . Since the action of γ is entirely controlled, we restrict the search to

braids in the positive monoid. In [GWY23], such a search strategy is used with the classical Garside structure, where braids are sampled in buckets indexed by the Garside length and the projective length (the spread of the matrices with the q -conventions from the beginning of the paper).

Our goal is to merge this search strategy with the ones based on curves inspired by Bigelow. This stands on the following lemma.

Lemma 6.2. *Let Γ be of simply-laced finite type. Fix a standard generator $\sigma_i \in B(\Gamma)$. Consider a writing $\beta = \beta_n \cdots \beta_1$ such that:*

- *the writing is a right-greedy normal form for β , that is: $\beta_i \in [1, \gamma]$ and for every $\mu \in \mathbb{T}$ such that $\mu \mid \beta_i$, we have $\mu\beta_{i-1} \notin [1, \gamma]$;*
- *$\beta\sigma_i$ is also a right-greedy form for its associated braid element;*
- *σ_i does not divide β_1 .*

Then $[\beta, \sigma_i] \neq \text{id}$.

Proof. Writing $[\beta, \sigma_i] = \beta\sigma_i\beta^{-1}\sigma_i^{-1}$ already gives a reduced minimal decomposition. $\square$

It should be emphasized that this writing for β is precisely designed to ensure that the categorical spread equals the Garside length: this follows from [LQ21, Prop 4.6 and Cor 4.10]. Then the non-triviality is stated in Proposition 4.11. We will use the previous lemma to find elements in the kernel of the Burau representation with coefficients in rings $F = \mathbb{Z}/p\mathbb{Z}[q, q^{-1}]$, as stated below.

Corollary 6.3. *Let F be a unital ring, and ϕ a unital ring homomorphism from $\mathbb{Z}[q, q^{-1}]$ to F . Assume that σ_i and β are as above, and furthermore that $\phi(\beta)\alpha_i = q^l\alpha_i$ (in $F \otimes_\phi V_q$). Then $[\beta, \sigma_i]$ is a non-trivial element in the kernel of the Burau representation with coefficients in F .*

Proof. The commutator $[\beta, \sigma] = (\beta\sigma_i\beta^{-1})\sigma_i^{-1}$ is a product of spherical twists over spherical objects that are not distinguished by the Burau representation with coefficients in F . We will check that the induced Burau action is trivial. The fact that the braid is non-trivial is a consequence of Theorem 6.2.

We will use an easy generalization of Eq. (1): for $\tau = \beta\sigma_i\beta^{-1}$ and $\alpha \in V_q$, we have:

$$\begin{aligned}\tau(\alpha) &= \alpha - \langle \beta\alpha_i, \alpha_d(\beta\alpha_i) \rangle \\ \tau^{-1}(\alpha) &= \alpha - q^{-1} \langle \alpha, \beta\alpha_i \rangle_d(\beta\alpha_i)\end{aligned}$$

The point of these formulas is to emphasize the role of a curve played by $\beta\alpha_i$ with respect to the twist $\tau = \beta\sigma_i\beta^{-1}$.

Write $[\beta, \sigma_i] = (\beta\sigma_i\beta^{-1})\sigma_i^{-1}$ and focus on $\beta\sigma_i\beta^{-1}$, whose action is thus given by:

$$\beta\sigma_i\beta^{-1}(\alpha) = \alpha - \langle \beta\alpha_i, \alpha \rangle_d(\beta\alpha_i).$$

Recall that $\beta\alpha_i = q^l\alpha_i$, so that:

$$\beta\sigma_i\beta^{-1}(\alpha) = \alpha - q^{-l} \langle \alpha_i, \alpha \rangle_d q^l \alpha_i = \alpha - \langle \alpha_i, \alpha \rangle_d \alpha_i.$$

This is the same action as σ_i , and thus $(\beta\sigma_i\beta^{-1})\sigma_i^{-1}$ has trivial Burau action. $\square$

This idea yielded several unfaithfulness results in type D_4 . Note that unfaithfulness over F_p for $p = 2, 3$ or 5 follows from type A embedding.

Theorem 6.4. *The Burau representation of type D_4 is not faithful over $\mathbb{Z}/p\mathbb{Z}$ with $p \leq 16$.*

Proof. Take $\sigma_1 = \sigma_1$ and β as follows. Rather than using the classical generators σ_i , in the interest of space, we will only provide the sequence of indices. i stands for σ_i , while $-i$ stands for σ_i^{-1} .

For $p = 6$, we take:

$$\begin{aligned}[-2, -1, 2, -4, -3, 1, -2, -1, 3, 4, -2, -1, -3, -4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -1, -3, -4, -3, -4, -2, 4, 3, -2, 1, 2, -3, \\ -2, -1, 1, 2, -4, -2, -1, -3, -4, -2, 4, 3, 2, -4, -3, 1, -2, -1, 3, 4, -2, -3, -4, -2, 4, 3, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4]\end{aligned}$$

For $p = 7$ we take:

$$\begin{aligned}[2, -3, -4, -2, 4, 3, 2, -3, -2, 2, -4, -3, 1, -2, -1, 3, 4, -2, -1, -3, -4, -2, 2, -3, -2, 1, 2, -4, -2, -1, -1, -3, 1, -2, -1, 2, -3, -2, 2, -4, \\ -2, -4, -3, 1, -2, -1, 3, 4, 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, -2, -1, -1, 2, -4, -3, 1, -2, -1, 3, 4, -2, -3, 1, -2, -1, 2, -3, -2, 2, -4, -2, \\ -4, -3, 1, -2, -1, 3, 4, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1]\end{aligned}$$

For $p = 8$, we take:

$$\begin{aligned}[-2, -1, -3, -4, -2, -1, -4, 2, -3, -2, 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, -2, -1, -3, -4, -2, 4, 3, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, \\ -3, -4, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, -3, -4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, \\ -1, -3, -4, -2, 4, 3, 2, -4, -3, 1, -2, -1, 3, 4, -2, -1, -3, 1, 2, -4, -2, -1]\end{aligned}$$

For $p = 9$, we find:

$$\begin{aligned} & [-2, -1, -3, -4, -2, 4, 3, -2, 2, -4, -3, 1, -2, -1, 3, 4, -2, -2, 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, -2, -1, -1, -4, -3, 1, -2, -1, 3, 4, 2, -3, \\ & -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, 1, -2, -1, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, 2, -4, -3, 1, -2, -1, 3, 4, -2, -1, -3, -4, 2, \\ & -4, -3, 1, -2, -1, 3, 4, -2, -1, -3, -4, -3, -4, -2, 4, 3, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -3, -4, -2, 4, 3, -2, 1, 2, -3, -2, -1, 1, 2, \\ & -4, -2, -1, 1, -2, -1, -3, -4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, 2, -3, -2, 2, -4, -2, -2, 1, 2, \\ & -3, -2, -1, 1, 2, -4, -2, -1] \end{aligned}$$

For $p = 10$ we find:

$$\begin{aligned} & [2, 3, -1, -4, -2, 2, -4, -3, 1, -2, -1, 3, 4, -2, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, 1, -2, -1, 2, -3, -2, 2, -4, -2, -4, \\ & -3, 1, -2, -1, 3, 4, 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, -3, -4, -2, 2, -4, -3, 1, -2, -1, 3, 4, \\ & -2, -1, -3, 1, 2, -4, -2, -1, -1, -3, 1, -2, -1, -1, -3, -4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -1, 1, -2, -1, -4, 2, \\ & -3, -2, 2, -4, -3, 1, -2, -1, 3, 4, -2] \end{aligned}$$

For $p = 11$, we take:

$$\begin{aligned} & [2, 1, 3, -4, 1, -2, -1, -1, 1, 2, -4, -2, -1, -1, 1, -2, -1, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -2, 2, -3, -2, 1, 2, -4, \\ & -2, -1, -3, 1, -2, -1, -3, -4, -2, 4, 3, 2, -4, -3, 1, -2, -1, 3, 4, -2, -3, 2, -3, -2, -2, 2, -3, -2, 2, -4, -2, -2, 2, -3, -2, 1, \\ & 2, -4, -2, -1, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, -3, -4, 2, -3, -2, -4, -3, 1, -2, -1, 3, 4, 1, 2, -3, -2, -1, -3, -4, \\ & -2, 4, 3, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -3, 1, 2, -4, -2, -1, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1] \end{aligned}$$

For $p = 12$ we find:

$$\begin{aligned} & [2, 1, 3, 4, 2, -3, -2, 2, -4, -2, 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, -2, -1, -1, -3, -4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, \\ & 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, -2, -1, -1, -3, -4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, 2, -4, -3, 1, -2, -1, 3, 4, -2, \\ & 1, -2, -1] \end{aligned}$$

For $p = 13$ we find:

$$\begin{aligned} & [-2, -1, -4, -3, -2, -1, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, -3, -2, 1, 2, -3, -2, -1, -1, -4, -3, 2, -4, -4, -2, -4, -3, 1, \\ & -2, -1, 3, 4, -1, -3, -4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -1, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -2, 1, 2, -3, -2, -1, 1, 2, \\ & -4, -2, -1, 1, -2, -1, -1, 1, 2, -4, -2, -1, 2, -3, -2, 1, 2, -4, -2, -1, 1, -2, -1, -4, 2, -3, -2, -4, 2, -3, -2, 2, -4, -3, 1, -2, -1, \\ & 3, 4, -2] \end{aligned}$$

For $p = 14$ we find:

$$\begin{aligned} & [-2, -1, -1, -3, -4, -4, 2, -3, -2, -4, 2, -3, -2, 2, -4, -3, 1, -2, -1, 3, 4, -2, -1, -3, -3, -4, -2, 4, 3, 2, -4, -3, 1, -2, -1, 3, 4, \\ & -2, 2, -4, -2, 1, 2, -3, -2, -1, -1, -3, -4, -4, 2, -3, -2, -1, 2, -3, -2, 2, -4, -2, -1, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, 1, \\ & -2, -1, -1, -3, -4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -4, -2, 1, 2, -3, -2, -1, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, \\ & 1, -2, -1, -4, 2, -3, -2, 2, -3, -2, 1, 2, -4, -2, -1, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -3, -4, 1, -2, -1, -1, -3, -4, 2, -4, \\ & -2, -4, -3, 1, -2, -1, 3, 4, -2, 1, 2, -4, -2, -1, -3, -3, 1, -2, -1, -3, -4, -2, 4, 3, 2, -4, -2, -3, -2, 1, 2, -4, -2, -1, -3, -2, 1, \\ & 2, -4, -2, -1, -3, -2, 1, 2, -4, -2, -1, -3, -2, 1, 2, -4, -2, -1, -4, 2, -3, -2, -1, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, -4, \\ & 2, -4, -2, -1, -3, -4, 2, -3, -2, 2, -4, -2, -1, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, -3, -4, 2, -4, -3, 1, -2, -1, 3, 4, -2, \\ & -4, 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, -2, -1, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, 1, -2, -1, -4, -3, 1, -2, -1, 3, 4, 1, 2, \\ & -3, -2, -1, -3, -4, -2, 4, 3, 2, -3, -2, -4, -3, 1, -2, -1, 3, 4, -1, -3, -4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -4, 1, -2, \\ & -1, -1, -3, -4, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -1, -3, 2, -4, -3, 1, -2, -1, 3, 4, -2] \end{aligned}$$

For $p = 15$ we find:

$$\begin{aligned} & [2, 3, 4, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -3, 1, -2, -1, -3, 1, -2, -1, -3, -4, -2, 4, 3, 1, 2, -4, -2, -1, -3, -4, -2, 4, 3, -1, -2, 1, 2, \\ & -3, -2, -1, 1, 2, -4, -2, -1, 1, -2, -1, -1, -3, -4, -3, -4, -2, 4, 3, -4, -3, 1, -2, -1, 3, 4, -4, -3, 1, -2, -1, 3, 4, -1, 2, -3, -2, 2, -4, \\ & -2, -1, -1, -3, -4, 2, -4, -3, 1, -2, -1, 3, 4, -2, -1, -3, -4, 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -4, -2, -1, \\ & -1, -3, -4, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -1, -3, 2, -4, -3, 1, -2, -1, 3, 4, -2] \end{aligned}$$

For $p = 16$ we find:

$[-2, -1, -3, -4, -2, 4, 3, -2, 2, -4, -3, 1, -2, -1, 3, 4, -2, -2, 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, -2, -1, 2, -3, -2, 2, -4, -2, -4, -3, 1,$
 $-2, -1, 3, 4, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, -4, -3, 1, -2, -1, 3, 4, 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, -2, -1, -1, -3, -4, 2, -3,$
 $-2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, -3, -4, -1, -3, -4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2,$
 $-1, 3, 4, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, -1, -3, -4, -3, -4, -2, 4, 3, 2, -4, -3, 1, -2, -1, 3, 4, -2, 1, -2, -1, -1, -4, -3, 1,$
 $-2, -1, 3, 4, -1, -4, -3, 1, -2, -1, 3, 4, 2, -3, -2, 2, -4, -2, -4, -3, 1, -2, -1, 3, 4, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1, 2, -3, -2, 2,$
 $-4, -2, -2, 1, 2, -3, -2, -1, 1, 2, -4, -2, -1]$

□

These results can be recovered by running the code available on [BQ24].

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