TORUS ACTIONS AND TENSOR PRODUCTS OF INTERSECTION COHOMOLOGY

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ABSTRACT. Given certain intersection cohomology sheaves on a projective variety with a torus action, we relate the cohomology groups of their tensor product to the cohomology groups of the individual sheaves. We also prove a similar result in the case of equivariant cohomology.

1. INTRODUCTION

Let X be a smooth complex projective variety together with an action of an algebraic torus T with isolated fixed points. We fix a regular algebraic one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$, which means that the set of λ-fixed points on X equals the set of T-fixed points on X (denoted X^T). Consider the Białynicki-Birula decomposition (see, e.g., [BB73]) of X: for each $w \in X^T$ define the *plus* and *minus* cells to be respectively

$$U_w = U_w^+ = \{x \in X \mid \lim_{t\to 0} \lambda(t)\cdot x = w\}, \quad t \in \mathbb{C}^*, \text{ and}$$
$$U_w^- = \{x \in X \mid \lim_{t\to \infty} \lambda(t)\cdot x = w\}, \quad t \in \mathbb{C}^*.$$

Each plus or minus cell is a λ-stable affine space, and hence the decompositions $X = \coprod_{w\in X^T} U_w$ and $X = \coprod_{w\in X^T} U_w^-$ are cell decompositions. For the purposes of this paper, we make the following additional assumptions on the T-action on X.

Assumption 1.1. The cell decompositions $X = \coprod_{w\in X^T} U_w$ and $X = \coprod_{w\in X^T} U_w^-$ are algebraic stratifications of X. In particular, the closure of every plus (resp. minus) cell is a union of plus (resp. minus) cells.

Assumption 1.2. For each $w \in X^T$, there is a one-parameter subgroup $\lambda_w\colon \mathbb{C}^* \to T$ and a neighbourhood V_w of w such that $\lim_{t\to 0} \lambda_w(t)\cdot v = w$ for every $v \in V_w$ and $t \in \mathbb{C}^*$.

Through most of this paper, we use the word *sheaf* to mean an object in $D^b_{c,\mathrm{BB}}(X,\mathbb{C})$, the bounded derived category of sheaves of $\mathbb{C}$-vector spaces on X that are constructible with respect to the Białynicki-Birula stratification. (Here we make use of Assumption 1.1.) Moreover all functors are derived, so for ease of notation we omit the decorations R and L.

For each $w \in X^T$, let IC_w denote the intersection cohomology sheaf on the closure of the cell U_w, extended by zero to all of X. The main theorem of the paper describes the cohomology of the tensor products of a collection of IC_w, in terms of the tensor products of the cohomologies of the individual IC_w.

1.1. **Main result.** Let $\Delta\colon X \to X^m$ be the diagonal embedding. Consider any sheaves $\mathscr{F}_1, \dots, \mathscr{F}_m$ in $D^b_{c,\mathrm{BB}}(X,\mathbb{C})$. Then their (derived) tensor product is also a sheaf in $D^b_{c,\mathrm{BB}}(X,\mathbb{C})$, and will be denoted by $\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_m$. Recall that

$$\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_m = \Delta^{-1}(\mathscr{F}_1 \boxtimes \cdots \boxtimes \mathscr{F}_m).$$

For any sheaf $\mathscr{F}$, its cohomology $H^\bullet(\mathscr{F}) = H^\bullet(X,\mathscr{F})$ is a graded vector space. There is a natural cup-product $\cup\colon H^\bullet(\mathscr{F}_1) \otimes \cdots \otimes H^\bullet(\mathscr{F}_m) \to H^\bullet(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_m)$, defined in Subsection 2.2.

Let $\underline{\mathbb{C}}$ denote the constant sheaf on X. For any sheaf $\mathscr{F}$, its cohomology $H^\bullet(\mathscr{F})$ is naturally a (graded) left and right module over the (graded) ring $H(X) = H^\bullet\left(X, \underline{\mathbb{C}}\right)$, as follows:

$$\cup\colon H(X) \otimes H^\bullet(\mathscr{F}) \to H^\bullet\left(\underline{\mathbb{C}} \otimes \mathscr{F}\right) \xrightarrow{\cong} H^\bullet(\mathscr{F}),$$
$$\cup\colon H^\bullet(\mathscr{F}) \otimes H(X) \to H^\bullet\left(\mathscr{F} \otimes \underline{\mathbb{C}}\right) \xrightarrow{\cong} H^\bullet(\mathscr{F}).$$

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Moreover, the cup-product descends to a morphism

$$H^\bullet(\mathscr{F}_1) \underset{H(X)}{\otimes} \cdots \underset{H(X)}{\otimes} H^\bullet(\mathscr{F}_m) \to H^\bullet(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_m).$$

Theorem 1.3. *Let $(p_1, \ldots, p_m)$ be an m-tuple of T-fixed points of X. If the assumptions 1.1 and 1.2 hold, then the cup-product map*

$$H^\bullet\left(\mathrm{IC}_{p_1}\right) \underset{H(X)}{\otimes} \cdots \underset{H(X)}{\otimes} H^\bullet\left(\mathrm{IC}_{p_m}\right) \to H^\bullet\left(\mathrm{IC}_{p_1} \otimes \cdots \otimes \mathrm{IC}_{p_m}\right) \tag{1}$$

is an isomorphism.

Since X is a T-space, each IC sheaf IC_{p_j} carries a canonical T-equivariant structure, and so does the tensor product $\mathrm{IC}_{p_1} \otimes \cdots \otimes \mathrm{IC}_{p_m}$. Let $H_T(X) = H_T^\bullet(X, \mathbb{C})$ be the T-equivariant cohomology of X. For any T-equivariant sheaf $\mathscr{F}$ on X, its T-equivariant cohomology $H_T^\bullet(\mathscr{F}) = H_T^\bullet(X, \mathscr{F})$ is a graded $H_T(X)$-module. As before, there is a cup-product map for T-equivariant cohomology, which factors through $H_T(X)$.

Theorem 1.4. *Under the assumptions 1.1 and 1.2, the cup-product map*

$$H_T^\bullet\left(\mathrm{IC}_{p_1}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H_T^\bullet\left(\mathrm{IC}_{p_m}\right) \to H_T^\bullet\left(\mathrm{IC}_{p_1} \otimes \cdots \otimes \mathrm{IC}_{p_m}\right)$$

is an isomorphism.

Remark 1.5. Even though our results are stated using IC sheaves, it is possible that they generalize to parity sheaves (defined and discussed by Juteau, Mautner, and Williamson in [JMW]). Our results and proof methods are similar to the main theorem from Ginzburg's paper [Gin91]. In [AR13, Theorem 4.1], Achar and Rider prove a version of Ginzburg's theorem for parity sheaves on generalized flag varieties of a Kac-Moody group. Similar generalizations may work in our case as well.

2. Setup

2.1. **The Białynicki-Birula stratification.** One can find (see, e.g. [Sum74] or [Kam66]) a T-equivariant projective embedding of X into some $\mathbb{P}^N$, such that the action of T on $\mathbb{P}^N$ is linear. Consider the following standard Morse-Bott function on $\mathbb{P}^N$:

$$[z_0 : \cdots : z_N] \mapsto \frac{\sum_{i=0}^N c_i |z_i|^2}{\sum_{i=0}^N |z_i|^2},$$

where c_i are the weights of the λ-action on $\mathbb{P}^N$. The critical sets of this function are precisely the T-fixed points on $\mathbb{P}^N$. The Morse-Bott cells of this function are locally closed algebraic subvarieties of $\mathbb{P}^N$. Since X has isolated T-fixed points, one can show that the composition $f : X \to \mathbb{P}^N \to \mathbb{R}$ is a Morse function with critical set X^T (see, e.g. [Aud04]). Each cell of the Morse decomposition under f is a preimage of a Morse-Bott cell of $\mathbb{P}^N$. Hence it is a locally closed algebraic subvariety of X. Moreover, each cell of the Morse decomposition is known to be a union of Białynicki-Birula plus-cells. A discussion of this may also be found [CG10, Section 2.4].

The collection of fixed points of the λ-action carries a partial order, where $v < w$ if $U_v \subset \overline{U_w}$. By the previous discussion, we see that $v < w$ iff $f(v) < f(w)$. Fix a weakly increasing enumeration $\{0, 1, \ldots, N\}$ of the points of X^T (sometimes denoted $\{w_0, \ldots, w_N\}$), and set $X_n = \bigcup_{i \le n} U_i$. Since the closure of every plus cell is a union of plus cells, it follows from the previous discussion that each X_n is a closed subvariety of X.

Similarly, set $X_n^- = \bigcup_{i \ge n} U_i^-$. By using the Morse function $(-f)$ instead of f, we see that each X_n^- is a closed subvariety of X. Hence we obtain two increasing filtrations of X by closed subvarieties: $X_0 \subset \cdots \subset X_N = X$ and $X_N^- \subset \cdots \subset X_0^- = X$.

We have the following inclusions:

$$X_n \overset{i_n}{\hookrightarrow} X, \quad X_{n-1} \overset{v}{\hookrightarrow} X_n \overset{u}{\hookleftarrow} U_n.$$

For any point $p \in X_n^-$, we have $f(w_n) \le f(p)$, with equality only if $p \in X^T$. For any point $p \in X_n$, we have $f(p) \le f(w_n)$, with equality only if $p \in X^T$. Hence if $p \in X_n^- \cap X_n$, then $f(p) = f(w_n)$, and $p \in X^T$. But $X_n^- \cap X_n \cap X^T = \{w_n\}$, and it follows that $p = w_n$. Hence for every n, the subvarieties X_n^- and X_n intersect transversally in the single point w_n.

Let $c_n \in H^\bullet(X)$ be the Poincaré dual to the homology class of X_n^-. As a vector space, $H^\bullet(X)$ is generated by the collection $\{c_n\}$. Finally, fix an m-tuple $(p_1, \ldots, p_m)$ of T-fixed points of X, and set $L_{j,n} = i_n^{-1} \mathrm{IC}_{p_j}$ for each j and n.

2.2. **The cup-product in cohomology.** Let $\pi\colon X \to \mathrm{pt}$ be the unique morphism to a point. For any sheaf $\mathscr{F}$ on X, its cohomology $H^\bullet(\mathscr{F})$ is a graded vector space, and may be thought of as $\pi_*\mathscr{F}$. We use this to define the cup-product map.

Recall that the functors (π^{-1}, π_*) form an adjoint pair, which has a counit $\pi^{-1}\circ\pi_* \to \mathrm{id}$. Let $\mathscr{F}_1, \dots, \mathscr{F}_m$ be sheaves on X. Tensoring the counit maps together, we have a map

$$\pi^{-1}\circ\pi_*(\mathscr{F}_1)\otimes\cdots\otimes\pi^{-1}\circ\pi_*(\mathscr{F}_m) \to \mathscr{F}_1\otimes\cdots\otimes\mathscr{F}_m\,.$$

The left hand side is canonically isomorphic to $\pi^{-1}(\pi_*\mathscr{F}_1\otimes\cdots\otimes\pi_*\mathscr{F}_m)$. Using the (π^{-1}, π_*) adjunction once more, we obtain the *cup-product*:

$$\cup\colon \pi_*\mathscr{F}_1\otimes\cdots\otimes\pi_*\mathscr{F}_m \to \pi_*(\mathscr{F}_1\otimes\cdots\otimes\mathscr{F}_m).$$

The cup-product gives each $H^\bullet(\mathscr{F}_i)$ the structure of a left and right module over $H(X)$. This module structure induces the following map, also called the cup-product:

$$H^\bullet(\mathscr{F}_1)\underset{H(X)}{\otimes}\cdots\underset{H(X)}{\otimes}H^\bullet(\mathscr{F}_m) \to H^\bullet(\mathscr{F}_1\otimes\cdots\otimes\mathscr{F}_m)\,.$$

Proposition 2.1. *For every n, the cup-product map*

$$H^\bullet\left(L_{1,n}\right)\underset{H(X)}{\otimes}\cdots\underset{H(X)}{\otimes}H^\bullet\left(L_{m,n}\right) \to H^\bullet\left(L_{1,n}\otimes\cdots\otimes L_{m,n}\right) \tag{2}$$

is an isomorphism.

When $X_n = X$, we have $L_{j,n} = \mathrm{IC}_{p_j}$ for each j. Hence Theorem 1.3 follows from this proposition, and we now focus on proving the proposition.

3. Proof of the isomorphism

We prove Proposition 2.1 by induction on the nth filtered piece of $X_0 \subset \cdots \subset X_N$. In the base case of $n = 0$, the space X_0 is zero-dimensional. Hence each sheaf $L_{j,0}$ is isomorphic to its cohomology. In this case the cup-product map (2) reduces to the identity map, which is an isomorphism.

Now we prove the induction step on the filtered piece X_n. We mainly use the following distinguished triangles:

$$u_!u^{-1}L_{j,n} \to L_{j,n} \to v_*v^{-1}L_{j,n}, \tag{3}$$

$$v_!v^!L_{j,n} \to L_{j,n} \to u_*u^{-1}L_{j,n}. \tag{4}$$

After taking cohomology, each of the above distinguished triangles produces a long exact sequence. In our case, all connecting homomorphisms of these long exact sequences vanish (see, e.g. [Soe90, Lemma 20] and [Gin91, Proposition 3.2]).

For brevity, we will use the following notation through the remainder of the paper.

$$\begin{aligned} M_{m,n} &= L_{2,n}\otimes\cdots\otimes L_{m,n}, \\ A_{m,n} &= H^\bullet\left(L_{2,n}\right)\underset{H(X)}{\otimes}\cdots\underset{H(X)}{\otimes}H^\bullet\left(L_{m,n}\right), \\ B_{m,n} &= H^\bullet\left(u_*u^{-1}L_{2,n}\right)\underset{H(X)}{\otimes}\cdots\underset{H(X)}{\otimes}H^\bullet\left(u_*u^{-1}L_{m,n}\right). \end{aligned} \tag{5}$$

The following two lemmas prove the proposition on the open part U_n in X_n.

Lemma 3.1. *Let $\mathscr{F}$ and $\mathscr{G}$ be any complexes of sheaves on U_n with locally constant cohomology sheaves. Then the cup-product map*

$$\cup\colon H^\bullet(u_!\mathscr{F})\otimes H^\bullet(u_*\mathscr{G}) \to H^\bullet(u_!\mathscr{F}\otimes u_*\mathscr{G})$$

is an isomorphism. Since $\cup$ factors through the surjection

$$H^\bullet(u_!\mathscr{F})\otimes H^\bullet(u_*\mathscr{G}) \twoheadrightarrow H^\bullet(u_!\mathscr{F})\underset{H(X)}{\otimes}H^\bullet(u_*\mathscr{G}),$$

the induced cup-product

$$\cup\colon H^\bullet(u_!\mathscr{F})\underset{H(X)}{\otimes}H^\bullet(u_*\mathscr{G}) \to H^\bullet(u_!\mathscr{F}\otimes u_*\mathscr{G})$$

is also an isomorphism.

Proof. Consider the following commutative diagram, where π is the projection to a point.

$$\begin{array}{ccc} U_n & \xrightarrow{u} & X_n \\ & \searrow^{p=\pi\circ u} & \downarrow \pi \\ & & \mathrm{pt} \end{array}$$

Recall that if A and B are any two complexes on X, then the cup-product is induced by adjunction from the natural map

$$\pi^{-1}(\pi_* A \otimes \pi_* B) \cong \pi^{-1}\pi_* A \otimes \pi^{-1}\pi_* B \to A \otimes B,$$

which may be broken up as follows:

$$\pi^{-1}\pi_* A \otimes \pi^{-1}\pi_* B \to A \otimes \pi^{-1}\pi_* B \to A \otimes B.$$

Therefore the cup-product map may be broken up as follows:

$$\pi_* A \otimes \pi_* B \to \pi_*(A \otimes \pi^{-1}\pi_* B) \to \pi_*(A \otimes B).$$

In our case, this becomes the following sequence of maps:

$$\pi_* u_! \mathscr{F} \otimes \pi_* u_* \mathscr{G} \xrightarrow{\mu_1} \pi_*(u_! \mathscr{F} \otimes \pi^{-1}\pi_* u_* \mathscr{G}) \xrightarrow{\mu_2} \pi_*(u_! \mathscr{F} \otimes u_* \mathscr{G}).$$

Since π is a proper map, we know that $\pi_* \cong \pi_!$, and hence μ_1 is an isomorphism by the projection formula. It remains to show that μ_2 is an isomorphism.

The pair of adjoint functors (π^{-1}, π_*) gives the counit morphism $p^{-1}p_* \mathscr{G} \to u^{-1}u_* \mathscr{G}$. The key observation is that this map is an isomorphism, because $\mathscr{G}$ is a direct sum of its cohomology sheaves on the affine space U_n. Now consider the following commutative diagram.

$$\begin{array}{ccc} u_! \mathscr{F} \otimes \pi^{-1}\pi_* u_* \mathscr{G} & \xrightarrow[\text{(proj.)}]{\cong} & u_!(\mathscr{F} \otimes p^{-1}p_* \mathscr{G}) \\ \mu_2 \downarrow \text{(counit)} & & \cong \downarrow \text{(counit)} \\ u_! \mathscr{F} \otimes u_* \mathscr{G} & \xrightarrow[\text{(proj.)}]{\cong} & u_!(\mathscr{F} \otimes u^{-1}u_* \mathscr{G}) \end{array} \tag{6}$$

The map μ_2 is obtained by applying the functor π_* to the left vertical map in (6) above. The diagram shows that this map is an isomorphism, and hence μ_2 is also an isomorphism. □

Lemma 3.2. *The cup-product map induces an isomorphism*

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} B_{m,n} \xrightarrow{\cong} H_c^\bullet\left(u^{-1}(L_{1,n} \otimes M_{m,n})\right).$$

Proof of lemma. Using Lemma 3.1 for $\mathscr{F} = u^{-1}L_{1,n}$ and $\mathscr{G} = u^{-1}L_{2,n}$, we obtain an isomorphism

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} H^\bullet\left(u_* u^{-1} L_{2,n}\right) \xrightarrow{\cong} H^\bullet\left(u_! u^{-1} L_{1,n} \otimes u_* u^{-1} L_{2,n}\right).$$

Moreover, we know that $u^{-1}u_* u^{-1} L_{2,n} \cong u^{-1}L_{2,n}$. Using this fact and the projection formula, we have

$$\begin{aligned} H^\bullet\left(u_! u^{-1} L_{1,n} \otimes u_* u^{-1} L_{2,n}\right) &\cong H^\bullet\left(u_!\left(u^{-1} L_{1,n} \otimes u^{-1} u_* u^{-1} L_{2,n}\right)\right) \\ &\cong H^\bullet\left(u_! u^{-1}(L_{1,n} \otimes L_{2,n})\right). \end{aligned}$$

All together, we get an isomorphism

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} H^\bullet\left(u_* u^{-1} L_{2,n}\right) \xrightarrow{\cong} H^\bullet\left(u_! u^{-1}(L_{1,n} \otimes L_{2,n})\right),$$

which can be written in our previously-introduced notation as

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} B_{2,n} \xrightarrow{\cong} H^\bullet\left(u_! u^{-1}(L_{1,n} \otimes M_{2,n})\right).$$

Now we can successively tensor the above map over $H(X)$ with the spaces $H^\bullet\left(u_* u^{-1} L_{i,n}\right)$, with i ranging from 3 to m. Each time, we apply Lemma 3.1 for $\mathscr{F} = u^{-1}(L_{1,n} \otimes M_{i-1,n})$ and $\mathscr{G} = u^{-1} L_{i,n}$ and use the argument above. Ultimately this construction yields

$$\begin{aligned} H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} B_{m,n} &\xrightarrow{\cong} H^\bullet\left(u_! u^{-1}(L_{1,n} \otimes M_{m-1,n})\right) \underset{H(X)}{\otimes} H^\bullet\left(u_* u^{-1} L_{m,n}\right) \\ &\xrightarrow{\cong} H^\bullet\left(u_!(u^{-1}(L_{1,n} \otimes M_{m,n}))\right) \\ &\cong H_c^\bullet\left(u^{-1}(L_{1,n} \otimes M_{m,n})\right). \end{aligned}$$

□

The next lemma is a refinement of a standard cohomology exact sequence to our particular case.

Lemma 3.3. *There is an exact sequence*

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} B_{m,n} \to H^\bullet\left(L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \to H^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \to 0.$$

Proof. Consider the distinguished triangle (3) for the sheaf $L_{1,n}$. Taking cohomology and applying the functor $(-) \underset{H(X)}{\otimes} A_{m,n}$, we obtain the right-exact sequence

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \xrightarrow{f} H^\bullet\left(L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \xrightarrow{g} H^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \to 0.$$

Using the distinguished triangles (4) for each of the sheaves $L_{j,n}$ for $j \geq 2$, we have surjective morphisms

$$H^\bullet\left(L_{j,n}\right) \twoheadrightarrow H^\bullet\left(u_* u^{-1} L_{j,n}\right).$$

Taking the tensor product of all of these along with $H^\bullet\left(u_! u^{-1} L_{1,n}\right)$, we obtain a surjective morphism

$$H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \overset{h}{\twoheadrightarrow} H^\bullet\left(u_! u^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} B_{m,n}.$$

We now show that the map f factors through the map h, by showing that $f(\ker h) = 0$. Since all boundary maps in the cohomology long exact sequence of the triangles (4) vanish, the following set generates $\ker h$:

$$\{a_1 \otimes a_2 \otimes \cdots \otimes a_n \mid a_j \in H^\bullet\left(v_* v^! L_{j,n}\right) \text{ for some } 2 \leq j \leq m\}.$$

Consider any element $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in \ker h$. Suppose that $a_j \in H^\bullet\left(v_* v^! L_{j,n}\right)$. Recall the following commutative diagram, which is the content of [Gin91, 3.8a].

$$\begin{array}{ccccc} H^\bullet\left(v_* v^! L_{j,n}\right) & \hookrightarrow & H^\bullet\left(L_{j,n}\right) & \twoheadrightarrow & H^\bullet\left(u^{-1} L_{j,n}\right) \\ & & \Big\downarrow c_n & & c_n \Big\downarrow \cong \\ & & H^\bullet\left(L_{j,n}\right) & \hookleftarrow & H_c^\bullet\left(u^{-1} L_{j,n}\right) \end{array}$$

From this diagram it follows that $c_n a_j = 0$, and that $a_1 \in c_n H^\bullet\left(L_{1,n}\right)$. Since all tensor products are over $H(X)$, the image of $h(a_1 \otimes \cdots \otimes a_n)$ under f must be zero. Therefore f factors through h, and we obtain the desired short exact sequence. □

Finally, we use the induction hypothesis to tackle the right side of the right-exact sequence from the previous lemma.

Lemma 3.4. *The cup-product map induces an isomorphism*

$$H^\bullet\left(v_* v^{-1} L_{1,n}\right) \underset{H(X)}{\otimes} A_{m,n} \xrightarrow{\cong} H^\bullet\left(L_{1,n-1} \otimes M_{m,n-1}\right).$$

Proof of lemma. The cup-product map on the left hand side is the following composition:

$$H^\bullet\left(v_*v^{-1}L_{1,n}\right)\underset{H(X)}{\otimes}A_{m,n}\to H^\bullet\left(v_*v^{-1}L_{1,n}\right)\underset{H(X)}{\otimes}H^\bullet\left(M_{m,n}\right)\to H^\bullet\left(v_*v^{-1}L_{1,n}\otimes M_{m,n}\right),$$

where the first map is the cup-product on the last $(m-1)$ factors, and the second map is the cup-product of the first factor with the rest. The projection formula also shows that

$$H^\bullet\left(v_*v^{-1}L_{1,n}\otimes M_{m,n}\right)\cong H^\bullet\left(v^{-1}L_{1,n}\otimes v^{-1}M_{m,n}\right)\cong H^\bullet\left(L_{1,n-1}\otimes M_{m,n-1}\right).$$

By induction on m, we may assume that the cup-product $A_{m,n}\to H^\bullet\left(M_{m,n}\right)$ is an isomorphism, and hence the first map above is an isomorphism. It remains to show that the following map is an isomorphism:

$$H^\bullet\left(v_*v^{-1}L_{1,n}\right)\underset{H(X)}{\otimes}H^\bullet\left(M_{m,n}\right)\to H^\bullet\left(v_*v^{-1}L_{1,n}\otimes M_{m,n}\right)$$

The element $c_n\in H$ acts on $H^\bullet\left(v_*L_{1,n-1}\right)$ by zero, since $L_{1,n-1}$ is supported on X_{n-1}. Recall from [Gin91] that the cokernel of c_n on $H^\bullet\left(M_{m,n}\right)$ is just $H^\bullet\left(M_{m,n-1}\right)$. Hence

$$H^\bullet\left(v_*v^{-1}L_{1,n}\right)\underset{H(X)}{\otimes}H^\bullet\left(M_{m,n}\right)\cong H^\bullet\left(L_{1,n-1}\right)\underset{H(X)}{\otimes}H^\bullet\left(M_{m,n-1}\right).$$

Hence the map above can be rewritten as the cup-product map

$$H^\bullet\left(L_{1,n-1}\right)\underset{H(X)}{\otimes}H^\bullet\left(M_{m,n-1}\right)\to H^\bullet\left(L_{1,n-1}\otimes M_{m,n-1}\right),$$

which is an isomorphism by the induction hypothesis. □

We now apply Saito's theory of mixed Hodge modules ([Sai90, Sai88]) to obtain another short exact sequence, as follows. Every IC-sheaf has the additional structure of a pure mixed Hodge module, which induces a mixed Hodge structure on tensor products of the $L_{i,n}$.

Lemma 3.5.

(i) The cohomology $H^\bullet\left(L_{1,n}\otimes M_{m,n}\right)$ is pure.
(ii) There is a short exact sequence

$$0\to H^\bullet_c\left(u^{-1}(L_{1,n}\otimes M_{m,n})\right)\to H^\bullet\left(L_{1,n}\otimes M_{m,n}\right)\to H^\bullet\left(L_{1,n-1}\otimes M_{m,n-1}\right)\to 0.$$

Proof. The proof is by induction on n. When $n=0$, we have $X_{-1}=\emptyset$ and $U=X_0$. The open inclusion u is the identity map, and the closed inclusion v is the zero map, hence (ii) is clear in the base case.

The set X_0 consists of a single, T-fixed point of X. Call this point w. By Assumption 1.2, there exists a neighborhood V_w of w and a one-parameter subgroup $\lambda_w\colon\mathbb{C}^*\to T$ that contracts V_w to w. Let i_w denote the inclusion of $\{w\}$ into the corresponding V_w. Let j_w denote the inclusion of V_w into X. By applying [Spr84, Corollary 1] or [Bra03, Lemma 6] to the sheaves $j_w^{-1}\,\mathrm{IC}_{p_i}$ for each i, we see that

$$H^\bullet\left(V_w,j_w^{-1}\,\mathrm{IC}_{p_i}\right)\cong H^\bullet\left(i_w^{-1}j_w^{-1}\,\mathrm{IC}_{p_i}\right)=H^\bullet(L_{i,0}).$$

The functor $H^\bullet(V_w,j_w^{-1}(-))$ weakly increases weights, while the functor $H^\bullet(i_w^{-1}j_w^{-1}(-))$ weakly decreases weights. Hence $H^\bullet(L_{i,0})$ is pure for each i. Taking the tensor product, we see that $H^\bullet(L_{1,0})\otimes\cdots\otimes H^\bullet(L_{m,0})$ is pure. Since w is a single point, we can naturally make the following identification:

$$H^\bullet(L_{1,0})\otimes\cdots\otimes H^\bullet(L_{m,0})\cong H^\bullet\left(L_{1,0}\otimes\cdots\otimes L_{m,0}\right)=H^\bullet\left(L_{1,0}\otimes M_{m,0}\right).$$

Hence $H^\bullet\left(L_{1,0}\otimes M_{m,0}\right)$ is pure, and (i) is proved in the base case. A similar argument has been used in Lemma 3.5 of [Gin91].

For the induction step, consider the distinguished triangle (3) for $L_{1,n}$. Apply the functor $\left(-\otimes L_{2,n}\otimes\cdots\otimes L_{m,n}\right)$, which may be written as $\left(-\otimes M_{m,n}\right)$ in the notation of (5). This yields the following distinguished triangle:

$$u_!u^{-1}L_{1,n}\otimes M_{m,n}\to L_{1,n}\otimes M_{m,n}\to v_*v^{-1}L_{1,n}\otimes M_{m,n}.$$

By a repeated application of the projection formula, we may write the first term of this triangle as

$$u_!u^{-1}L_{1,n}\otimes M_{m,n}\cong u_!\left(u^{-1}L_{1,n}\otimes\cdots\otimes u^{-1}L_{m,n}\right)=u_!u^{-1}\left(L_{1,n}\otimes M_{m,n}\right),$$

and the third term of this triangle as

$$v_* v^{-1} L_{1,n} \otimes M_{m,n} \cong v_* \left(v^{-1} L_{1,n} \otimes \cdots \otimes v^{-1} L_{m,n} \right) = v_* \left(L_{1,n-1} \otimes M_{m,n-1} \right).$$

Taking cohomology, we obtain the following long exact sequence:

$$\cdots \to H_c^\bullet \left(u^{-1}(L_{1,n} \otimes M_{m,n}) \right) \to H^\bullet \left(L_{1,n} \otimes M_{m,n} \right) \to H^\bullet \left(L_{1,n-1} \otimes M_{m,n-1} \right) \to \cdots .$$

The term $H^\bullet \left(L_{1,n-1} \otimes M_{m,n-1} \right)$ is pure by the induction hypothesis.

From Lemma 3.2, we know that

$$H_c^\bullet \left(u^{-1} \left(L_{1,n} \otimes M_{m,n} \right) \right) \cong H_c^\bullet(u^{-1} L_{1,n}) \underset{H(X)}{\otimes} H^\bullet(u^{-1} L_{2,n}) \underset{H(X)}{\otimes} \cdots \underset{H(X)}{\otimes} H^\bullet(u^{-1} L_{m,n}).$$

Recall that U_n is the Białynicki-Birula plus-cell for the fixed point w_n. Hence the λ-action contracts U_n to w_n. By [Spr84, Corollary 2], we know that $H_c^\bullet \left(u^{-1} L_{1,n} \right)$ is isomorphic to the costalk of $u^{-1} L_{1,n}$ at w_n, which is isomorphic to a shift of the stalk of IC_{p_1} at w_n. For any $i > 1$, we know by [Spr84, Corollary 1] that $H^\bullet \left(u^{-1} L_{i,n} \right)$ is isomorphic to the stalk of $u^{-1} L_{i,n}$ at w_n, which is equal to the stalk of IC_{p_i} at w_n. By using Assumption 1.2 and the argument used earlier in this proof, we know that the stalk of each IC_{p_i} at any T-fixed point is pure, and hence the spaces $H_c^\bullet \left(u^{-1} L_{1,n} \right)$ as well as $H^\bullet \left(u^{-1} L_{i,n} \right)$ for $i > 1$ are all pure. Therefore the tensor product $H_c^\bullet \left(u^{-1} \left(L_{1,n} \otimes M_{m,n} \right) \right)$ is pure.

Since the terms on either side of the long exact sequence are pure, the connecting homomorphisms are zero, and hence $H^\bullet \left(L_{1,n} \otimes M_{m,n} \right)$ is also pure. This argument completes the induction step, and hence completes the proof. □

Putting together the exact sequences from Lemma 3.3 and Lemma 3.5, we obtain the following commutative diagram, where the vertical maps are induced by cup-products. In particular, the middle map b is just the map from Proposition 2.1.

$$\begin{array}{ccccccccc} & & H^\bullet \left(u_! u^{-1} L_{1,n} \right) \underset{H(X)}{\otimes} B_{m,n} & \longrightarrow & H^\bullet \left(L_{1,n} \right) \underset{H(X)}{\otimes} A_{m,n} & \longrightarrow & H^\bullet \left(v_* v^{-1} L_{1,n} \right) \underset{H(X)}{\otimes} A_{m,n} & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & H_c^\bullet \left(u^{-1}(L_{1,n} \otimes M_{m,n}) \right) & \longrightarrow & H^\bullet \left(L_{1,n} \otimes M_{m,n} \right) & \longrightarrow & H^\bullet \left(L_{1,n-1} \otimes M_{m,n-1} \right) & \longrightarrow & 0 \end{array} \tag{7}$$

The leftmost map a is an isomorphism by Lemma 3.2. The rightmost map c is an isomorphism by Lemma 3.4. By the snake lemma, the middle map b is an isomorphism as well, and Proposition 2.1 is proved.

4. Computation of equivariant cohomology

Consider a smooth complex projective variety X with the same assumptions as in Section 1. The goal of this section is to prove Theorem 1.4.

First we recall some constructions in equivariant cohomology. The main references are [BL94] and [GKM98]. Fix a universal principal T-bundle $ET \to BT$, where ET (respectively BT) is the direct limit over m of algebraic approximations ET_m (respectively BT_m). Consider the following diagram, where the map p is the second projection, and the map q is the quotient by the diagonal T-action.

$$\begin{array}{ccc} & ET \times X & \\ \swarrow^{p} & & \searrow^{q} \\ X & & ET \times_T X \end{array}$$

Since each stratum U_n is a locally closed T-invariant affine subvariety of X, the trivial local system on U_n gives rise to a canonically-defined sheaf $\overline{\mathrm{IC}_n}$ on $ET \times_T X$, and a canonical isomorphism $\beta : p^{-1}\, \mathrm{IC}_n \xrightarrow{\cong} q^{-1} \overline{\mathrm{IC}_n}$ (see, e.g., [BL94]). The triple $(\mathrm{IC}_n, \overline{\mathrm{IC}_n}, \beta)$ is called the equivariant IC sheaf corresponding to U_n.

4.1. **Equivariant homology and cohomology.** For any variety Y equipped with a T-action, the cohomology of $ET \times_T Y$ is called the *equivariant cohomology* of Y, and is denoted by $H_T^\bullet(Y)$. In particular, since $ET \times_T \mathrm{pt} \cong BT$, we have $H_T^\bullet(\mathrm{pt}) \cong H^\bullet(BT)$. The space $H_T^\bullet(Y)$ is a ring under cup-product, and is also an $H_T(X)$-module via pullback under the projection $Y \to \mathrm{pt}$. For convenience, we will denote $H_T^\bullet(X)$ by $H_T(X)$. In our case, $H_T(X)$ is isomorphic to $H^\bullet(X) \otimes H^\bullet(BT)$ as an $H_T(X)$-module (see, e.g., [GKM98, Theorem 14.1]). Similarly, the equivariant cohomology of any T-equivariant sheaf on X also carries an $H_T(X)$-module structure.

One can define the T-equivariant Borel-Moore homology of X, denoted $H_\bullet^T(X)$. Every T-equivariant closed subvariety Y of X defines a class $[Y]_T$ of degree $2\dim_{\mathbb{C}} Y$ in $H_\bullet^T(X)$. If X is smooth, then every class $[Y]_T$ has an equivariant Poincaré dual cohomology class in $H_T^\bullet(X)$. More details can be found in [Gra01] and [Bri00].

4.2. **Proof of the equivariant case.** Consider an m-tuple $(p_1, \ldots, p_m)$ of T-fixed points of X. Then $\mathrm{IC}_{p_1}, \ldots, \mathrm{IC}_{p_m}$ are the IC sheaves corresponding to $U_{p_1}, \ldots, U_{p_m}$ respectively. Let $L_{j,n} = i_n^{-1} \mathrm{IC}_{p_j}$ for each j and n.

Proposition 4.1. *Under the assumptions 1.1 and 1.2, the cup-product maps*

$$H_T^\bullet(L_{1,n}) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H_T^\bullet(L_{m,n}) \to H_T^\bullet(L_{1,n} \otimes \cdots \otimes L_{m,n})$$

are isomorphisms for each n.

When $X_n = X$, we have $L_{j,n} = \mathrm{IC}_{p_j}$ for each j. Hence this proposition implies Theorem 1.4. To prove the proposition, we first state two general lemmas about T-equivariant cohomology of sheaves.

Lemma 4.2. *Consider the fiber bundle $ET \times_T X \to BT$, with fiber X. Let IC_w be the (T-equivariant) IC sheaf on the closure of a stratum X_w, extended by zero to all of X. Then the Leray spectral sequence for the computation of $H_T^\bullet(X; \mathrm{IC}_w) = H^\bullet\left(ET \times_T X; \overline{\mathrm{IC}_w}\right)$ collapses at the E_2-page. Hence $H_T^\bullet(\mathrm{IC}_w)$ is isomorphic to $H^\bullet(\mathrm{IC}_w) \otimes H^\bullet(BT)$ as a graded $H^\bullet(BT)$-module.*

Proof. See [GKM98, Theorem 14.1]. The proof uses the fact that the cohomology of $BT \cong (\mathbb{C}\mathrm{P}^\infty)^{\dim T}$ is pure. □

Lemma 4.3. *Let Y be any T-space, and let $\mathscr{F}$ be a T-equivariant sheaf on Y such that the space $H^\bullet(Y; \mathscr{F})$ is pure. Then $H_T^\bullet(Y; \mathscr{F})$ is pure as well.*

Proof. Recall that $H_T^\bullet(Y, \mathscr{F}) = H^\bullet\left(ET \times_T X, \overline{\mathscr{F}}\right)$. The result follows from computing the Leray spectral sequence for the fiber bundle $ET \times_T Y \to BT$, and by using that $H^\bullet(BT)$ and $H^\bullet(Y, \mathscr{F})$ are pure. □

We also record some equivariant analogues of results stated in Section 3. First note that the boundary maps in the long exact sequences of T-equivariant cohomology for the distinguished triangles (3) and (4) vanish. The proof is analogous to the non-equivariant case, using Lemma 4.3.

The following lemma is an analogue of Lemma 3.1.

Lemma 4.4. *Let $U = X_n \backslash X_{n-1}$. Let $\mathscr{F}$ and $\mathscr{G}$ be any T-equivariant complexes of sheaves on U. Then the cup-product map*

$$\cup \colon H_T^\bullet(u_! \mathscr{F}) \underset{H^\bullet(BT)}{\otimes} H_T^\bullet(u_* \mathscr{G}) \to H_T^\bullet(u_! \mathscr{F} \otimes u_* \mathscr{G})$$

is an isomorphism. Since $\cup$ factors through the surjection

$$H_T^\bullet(u_! \mathscr{F}) \underset{H^\bullet(BT)}{\otimes} H_T^\bullet(u_* \mathscr{G}) \twoheadrightarrow H_T^\bullet(u_! \mathscr{F}) \underset{H_T(X)}{\otimes} H_T^\bullet(u_* \mathscr{G}),$$

the induced cup-product

$$H_T^\bullet(u_! \mathscr{F}) \underset{H_T(X)}{\otimes} H_T^\bullet(u_* \mathscr{G}) \to H_T^\bullet(u_! \mathscr{F} \otimes u_* \mathscr{G})$$

is also an isomorphism.

Proof. Consider the fiber bundle $ET \times_T X_n \to BT$, with fiber X_n. The E_2 pages of the Leray spectral sequences for $u_! \mathscr{F}$ and $u_* \mathscr{G}$ are as follows:

$$H^p(BT, H^q(u_! \mathscr{F})) \implies H_T^{p+q}(u_! \mathscr{F}),$$
$$H^r(BT, H^s(u_* \mathscr{G})) \implies H_T^{r+s}(u_* \mathscr{G}).$$

On the E_2 page, the cup-product map can be written as the composition of the following two maps. The first map is the cup-product with local coefficients, and the second is the fiber-wise cup-product on the local systems.

$$H^p(BT, H^q(u_! \mathscr{F})) \underset{H^\bullet(BT)}{\otimes} H^r(BT, H^s(u_* \mathscr{G})) \to H^{p+r}(BT, H^q(u_! \mathscr{F}) \otimes H^s(u_* \mathscr{G})),$$

$$H^{p+r}(BT, H^q(u_! \mathscr{F}) \otimes H^s(u_* \mathscr{G})) \to H^{p+r}(BT, H^{q+s}(u_! \mathscr{F} \otimes u_* \mathscr{G})).$$

Since the local systems $H^q(u_! \mathscr{F})$ and $H^s(u_* \mathscr{G})$ are constant on BT, the first map yields isomorphisms

$$H^\bullet(BT, H^q(u_! \mathscr{F})) \underset{H^\bullet(BT)}{\otimes} H^\bullet(BT, H^s(u_* \mathscr{G})) \xrightarrow{\cong} H^\bullet(BT, H^q(u_! \mathscr{F}) \otimes H^s(u_* \mathscr{G})).$$

Finally, we know from Lemma 3.1 that $H^\bullet(u_! \mathscr{F}) \otimes H^\bullet(u_* \mathscr{G}) \xrightarrow{\cong} H^\bullet(u_! \mathscr{F} \otimes u_* \mathscr{G})$ via the cup-product map. Altogether, the cup-product maps on the E_2 page yield an isomorphism

$$H^\bullet(BT, H^\bullet(u_! \mathscr{F})) \underset{H^\bullet(BT)}{\otimes} H^\bullet(BT, H^\bullet(u_* \mathscr{G})) \xrightarrow{\cong} H^\bullet(BT, H^\bullet(u_! \mathscr{F} \otimes u_* \mathscr{G})).$$

The left hand side is a tensor product of two free $H^\bullet(BT)$-modules over $H^\bullet(BT)$. Hence it converges to $H^\bullet_T(u_! \mathscr{F}) \underset{H^\bullet(BT)}{\otimes} H^\bullet_T(u_* \mathscr{G})$. The right hand side converges to $H^\bullet_T(u_! \mathscr{F} \otimes u_* \mathscr{G})$. Since the E_2 pages of the left hand side and the right hand side are isomorphic via the cup-product map, the following cup-product map

$$H^\bullet_T(u_! \mathscr{F}) \underset{H^\bullet(BT)}{\otimes} H^\bullet_T(u_* \mathscr{G}) \to H^\bullet_T(u_! \mathscr{F} \otimes u_* \mathscr{G})$$

is an isomorphism. □

Let $\widetilde{c_n} \in H_T(X)$ be the equivariant Poincaré dual of $[X_n^-]_T$. Each $\widetilde{c_n}$ restricts to the class c_n under the map $H_T(X) \to H^\bullet(X)$, hence the collection $\{\widetilde{c_n}\}$ generates $H_T(X)$ over $H^\bullet(BT)$.

The following lemma (analogous to [Gin91, 3.8a]) describes the action of $\widetilde{c_n}$ on the equivariant cohomology of the sheaves $L_{j,n}$ on X.

Lemma 4.5. *For every j, the action of $\widetilde{c_n}$ on $H^\bullet_T(L_{j,n})$ fits into the following commutative diagram:*

$$\begin{array}{ccc} H^\bullet_T(L_{j,n}) & \twoheadrightarrow & H^\bullet_T(u^{-1}L_{j,n}) \\ \downarrow \widetilde{c_n} & & \widetilde{c_n} \downarrow \cong \\ H^\bullet_T(L_{j,n}) & \hookleftarrow & H^\bullet_{T,c}(u^{-1}L_{j,n}) \end{array}$$

Proof. Recall that the intersection of X_n and X_n^- lies away from X_{n-1}. Hence $\widetilde{c_n}$ restricts to zero on X_{n-1}, and cup-product by $\widetilde{c_n}$ annihilates the cohomology of any sheaf supported on X_{n-1}. The kernel of $H^\bullet_T(L_{j,n}) \twoheadrightarrow H^\bullet_T(u^{-1}L_{j,n})$ and the cokernel of $H^\bullet_{T,c}(u^{-1}L_{j,n}) \to H^\bullet_T(L_{j,n})$ are both supported on X_{n-1}. So the map of multiplication by $\widetilde{c_n}$ from $H^\bullet_T(X_n)$ to $H^\bullet_T(X_n)$ factors as follows.

$$\begin{array}{ccc} H^\bullet_T(L_{j,n}) & \twoheadrightarrow & H^\bullet_T(u^{-1}L_{j,n}) \\ \downarrow \widetilde{c_n} & & \widetilde{c_n} \downarrow \\ H^\bullet_T(L_{j,n}) & \hookleftarrow & H^\bullet_{T,c}(u^{-1}L_{j,n}) \end{array}$$

It remains to show that the vertical map on the right is an isomorphism. Since X_n and X_n^- intersect transversally in the single point w_n, the restriction of $\widetilde{c_n}$ to X_n is the image in $H^\bullet_T(X_n)$ of a generator of the local cohomology group $H^\bullet_T(X_n, X_n \backslash \{w_n\})$.

Since $w_n \in U_n$, we have $H^\bullet_T(X_n, X_n \backslash \{w_n\}) \cong H^\bullet_T(U_n, U_n \backslash \{w_n\})$ by excision. But U_n is an affine space that is T-equivariantly contractible to w_n, and hence $H^\bullet_T(U_n, U_n \backslash \{w_n\}) \cong H^\bullet_{T,c}(U_n)$. This shows that multiplication by $\widetilde{c_n}$ maps $H^\bullet_T(U_n)$ isomorphically to $H^\bullet_{T,c}(U_n)$.

Since $u^{-1}L_{j,n}$ is T-equivariant, the above argument applies to the cohomology of $u^{-1}L_{j,n}$ as well. This means that $\widetilde{c_n}$ maps $H^\bullet_T(u^{-1}L_{j,n})$ isomorphically to $H^\bullet_{T,c}(u^{-1}L_{j,n})$, and the proof is complete. □

Once again, let $M_{m,n}$ denote the sheaf $L_{2,n} \otimes \cdots \otimes L_{m,n}$. For brevity, we set up the following additional notation.

$$\overline{A}_{m,n} = H^\bullet_T\left(L_{2,n}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H^\bullet_T\left(L_{m,n}\right),$$
$$\overline{B}_{m,n} = H^\bullet_T\left(u_* u^{-1} L_{2,n}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H^\bullet_T\left(u_* u^{-1} L_{m,n}\right).$$

The following two lemmas are analogues of Lemma 3.3 and Lemma 3.5 respectively.

Lemma 4.6. *There is an exact sequence*

$$H^\bullet_T\left(u_! u^{-1} L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{B}_{m,n} \to H^\bullet_T\left(L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{A}_{m,n} \to H^\bullet_T\left(v_* v^{-1} L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{A}_{m,n} \to 0.$$

Proof. The proof is analogous to the proof of Lemma 3.3. We use the fact that $H^\bullet_T(X) \cong H^\bullet(X) \otimes H^\bullet(BT)$, and use Lemma 4.5 as a substitute for [Gin91, 3.8a]. ☐

Lemma 4.7.

(i) The cohomology $H^\bullet_T\left(L_{1,n} \otimes M_{m,n}\right)$ is pure.

(ii) There is a short exact sequence

$$0 \to H^\bullet_{T,c}\left(u^{-1}(L_{1,n} \otimes M_{m,n})\right) \to H^\bullet_T\left(L_{1,n} \otimes M_{m,n}\right) \to H^\bullet_T\left(L_{1,n-1} \otimes M_{m,n-1}\right) \to 0.$$

Proof. The proofs are analogous to the proofs of their counterparts from Section 3, using the observation of Lemma 4.3 and the fact that $H^\bullet(BT)$ is pure. ☐

We now complete the proof of Theorem 1.4.

Proof of Theorem 1.4. We obtain the following commutative diagram from the exact sequences of Lemma 4.6 and Lemma 4.7.

$$\begin{array}{ccccccccc}
 & & H^\bullet_T\left(u_! u^{-1} L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{B}_{m,n} & \longrightarrow & H^\bullet_T\left(L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{A}_{m,n} & \longrightarrow & H^\bullet_T\left(v_* v^{-1} L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{A}_{m,n} & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & H^\bullet_T\left(u_! u^{-1} L_{1,n} \otimes M_{m,n}\right) & \longrightarrow & H^\bullet_T\left(L_{1,n} \otimes M_{m,n}\right) & \longrightarrow & H^\bullet_T\left(v_* v^{-1} L_{1,n} \otimes M_{m,n}\right) & \longrightarrow & 0
\end{array} \tag{8}$$

First observe that the action of $H_T(X)$ on $H^\bullet_T\left(u_! u^{-1} L_{1,n}\right)$ and on $\overline{B}_{m,n}$ factors through the map $H_T(X) \to H^\bullet_T(U) \cong H^\bullet(BT)$, so

$$H^\bullet_T\left(u_! u^{-1} L_{1,n}\right) \underset{H_T(X)}{\otimes} \overline{B}_{m,n} \cong H^\bullet_T\left(u_! u^{-1} L_{1,n}\right) \underset{H^\bullet(BT)}{\otimes} \overline{B}_{m,n}.$$

We prove by induction on m that the map a is an isomorphism. As in the proof of Lemma 3.2, the case of $m = 2$ is proved by Lemma 4.4, and the general case is proved by iterating the argument. An argument similar to the proof of Lemma 3.4 proves that the map c is an isomorphism.

Hence by the snake lemma, the middle map b is an isomorphism as well. Consequently, we obtain the following isomorphisms for every n:

$$H^\bullet_T\left(L_{1,n}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H^\bullet_T\left(L_{m,n}\right) \to H^\bullet_T\left(L_{1,n} \otimes \cdots \otimes L_{m,n}\right).$$

In particular when $X_n = X$, we see that the cup-product map

$$H^\bullet_T\left(\mathrm{IC}_{p_1}\right) \underset{H_T(X)}{\otimes} \cdots \underset{H_T(X)}{\otimes} H^\bullet_T\left(\mathrm{IC}_{p_m}\right) \to H^\bullet_T\left(\mathrm{IC}_{p_1} \otimes \cdots \otimes \mathrm{IC}_{p_m}\right)$$

is an isomorphism. ☐

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