EQUIVALENT COHOMOLOGY AND THE LOCALIZATION THEOREM

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1. Introduction

In this paper we describe equivariant cohomology, which is a cohomology theory applicable to spaces equipped with a group action. The equivariant cohomology measures not only the topology of the space, but also the complexity of the group action. For example if the group acts freely, the equivariant cohomology is equal to the ordinary cohomology of the orbit space. In contrast, the equivariant cohomology of a one-point space (on which any group acts trivially) is quite large.

Under certain conditions the equivariant cohomology ring may be described to a large extent by its restriction to the fixed points of the group action. This is the content of the localization theorem. We discuss versions by Atiyah-Bott ([AB84]) and Berline-Vergne ([BV85]) in the context of actions of compact connected Lie groups on compact smooth manifolds. As a consequence of the localization theorem, the integral of an equivariant cohomology class over a manifold can be expressed by an integral over just the fixed set of the group action, which is typically easier to compute.

In Section 2 and Section 3, we formally describe the construction of equivariant cohomology, and some complexes that are convenient for computing it. In Section 4 we describe how some properties of ordinary cohomology generalize to equivariant cohomology. In Section 5, we reduce to the case that the group is a compact torus, which is useful for computational purposes. Finally in Section 6, we precisely formulate the localization theorems and deduce the integration formula.

The main references are [GS99], [AB84] and [BV85].

2. Topological construction of equivariant cohomology

Throughout this paper, let G denote a compact connected Lie group. For any real vector space V, we denote its complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ by $V_{\mathbb{C}}$. Set $\mathfrak{g} = (\operatorname{Lie} G)_{\mathbb{C}}$. Let M be a topological space equipped with a G-action.

The main ingredient of the topological construction of equivariant cohomology is the *homotopy quotient*. Fix a classifying space BG of G, and a corresponding universal bundle $EG \to BG$. Observe that the diagonal action of G on the space $EG \times M$ is free.

Definition 2.1. The *homotopy quotient* of M by G is defined to be $M_G = EG \times_G M$. The *G-equivariant cohomology* of M is defined to be the singular cohomology of M_G, and is denoted by $H^*_G(M)$.

We always fix the ring of coefficients to be $\mathbb{C}$. As stated, the definition of $H^*_G(M)$ depends on a choice of EG and BG. However, it can be checked that the space M_G is well-defined upto homotopy type, and hence $H^*_G(M)$ is well-defined upto isomorphism.

Example 2.2. Let $M = \{p\}$ be a single point with the trivial G-action. Then $EG \times_G \{p\} \cong EG/G \cong BG$. Hence $H^*_G(\{p\}) = H^*(BG)$.

Example 2.3. Let M be a space with a free G-action. In this case, the map $EG \times_G M \to M/G$ is a fibre bundle with fibre E. Since E is contractible, we have $H^*_G(M) \cong H^*(M/G)$.

Following the notation of [AB84], we denote $H^*_G(\{p\})$ by H^*_G. For every M, there is a projection map $M_G \to BG$, which realizes M_G as a fibre bundle over BG with fibre M. Hence we obtain a map $H^*_G \to H^*_G(M)$ on cohomology. This makes $H^*_G(M)$ into an H^*_G-algebra (in particular, an H^*_G-module).

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3. Construction of equivariant de Rham cohomology

The de Rham complex is a convenient tool to compute the cohomology of smooth manifolds. We now describe analogues of the de Rham complex for the equivariant case. In this section let M be a smooth manifold equipped with a smooth G-action. Let $\Omega(M)$ be the usual de Rham complex of smooth differential forms on M.

3.1. **G^*-algebras.** To describe equivariant de Rham cohomology, we need the algebraic structure of G^* algebras, as defined in [GS99]. Before recalling the formal definition, we consider the motivating example of $\Omega(M)$. This is a commutative differential graded algebra (DGA) on which G acts by automorphisms. Explicitly, there is a smooth representation $\rho\colon G \to \mathrm{Aut}(\Omega(M))$ given by $\rho(g)(\omega) = (g^{-1})^*\omega$. By differentiation, we obtain a representation $L_\xi\colon \mathfrak{g} \to \mathrm{End}(\Omega(M))$. Every $\xi \in \mathfrak{g}$ determines a vector field v_ξ on M as follows:

$$v_\xi(x) = \frac{d}{dt}(\exp -t\xi)(x)\Big|_{t=0}. \tag{1}$$

Observe that L_ξ is just the Lie derivative with respect to v_ξ, which is a derivation of degree 0 on $\Omega(M)$. Let $\iota_\xi\colon \Omega^*(M) \to \Omega^{*-1}(M)$ denote contraction by v_ξ, which is a derivation of degree -1 on $\Omega(M)$. The operators L_ξ, ι_ξ, d and ρ satisfy several relations. This structure is encoded in the definition of a G^* algebra.

Definition 3.1. Let (A, d) be a commutative DGA over $\mathbb{C}$ with a smooth representation $\rho\colon G \to \mathrm{Aut}(A)$. Then A is called a G^* *algebra* if for every $\xi \in \mathfrak{g}$, there are derivations $\iota_\xi \in \mathrm{Der}_{-1}(A)$ and $L_\xi \in \mathrm{Der}_0(A)$ satisfying the following properties.

(1) $[L_\xi, L_\eta] = L_{[\xi,\eta]}$.
(2) $[\iota_\xi, \iota_\eta] = 0$.
(3) $[L_\xi, \iota_\eta] = \iota_{[\xi,\eta]}$.
(4) $[d, \iota_\xi] = L_\xi$.
(5) $[d, L_\xi] = 0$.
(6) $\frac{d}{dt}\rho(\exp t\xi)\,|_{t=0}= L_\xi$.
(7) $\rho_g \circ L_\xi \circ \rho_{g^{-1}} = L_{\mathrm{Ad}\,g(\xi)}$.
(8) $\rho_g \circ \iota_\xi \circ \rho_{g^{-1}} = \iota_{\mathrm{Ad}\,g(\xi)}$.
(9) $\rho_g \circ d \circ \rho_{g^{-1}} = d$.

Definition 3.2. An element a in a G^* algebra A is called *horizontal* if $\iota_\xi a = 0$ for every $\xi \in \mathfrak{g}$. A horizontal element $a \in A$ such that $\rho_g a = a$ for every $g \in G$ is called *basic.*

Let A_{hor} and A_{bas} denote the subsets of horizontal and basic elements of A respectively. Then both these spaces are subalgebras that are preserved by d. It is easy to check that if (A_1, d_1) and (A_2, d_2) are two G^* algebras, then $A_1 \otimes A_2$ is also a G^* algebra, where the maps ι_ξ and L_ξ are extended to $A_1 \otimes A_2$ as derivations in the obvious way.

Definition 3.3. Let $\xi_1, \ldots, \xi_n$ be a basis of $\mathfrak{g}$. A G^* algebra A is said to be of type (C) if there are elements $\{\theta^i \in A_1 \mid 1 \le i \le n\}$ such that $\iota_{\xi_i}\theta^j = \delta_{ij}$ for all i, j, and such that the subspace spanned by these elements is G-invariant.

Fix E to be any acyclic G^* algebra of type (C).

Definition 3.4. The G-equivariant cohomology of a G^* algebra A is defined to be the cohomology of the complex $(A \otimes E)_{\mathrm{bas}}$, and is denoted by $H^*_G(A)$.

The complex $(A \otimes E)_{\mathrm{bas}}$ can be compared with the homotopy quotient, where E may be thought of as playing the role of EG. The algebra E being acyclic corresponds to EG being contractible, and E being of type (C) corresponds to EG having a free G-action. As stated, the definition depends on the choice of E. It is proved in [GS99] that the definition is independent of choices. Moreover, the following theorem states the relationship of the G-equivariant cohomology of a manifold M (defined in the previous section) to the G-equivariant cohomology of $\Omega(M)$.

Theorem 3.5 (Equivariant de Rham theorem)**.** *The G-equivariant cohomology of M is isomorphic to the G-equivariant cohomology of* $\Omega(M)$.

3.2. **The Weil algebra and the Cartan complex.** We now introduce two specific complexes that compute the equivariant cohomology of a smooth manifold. We state the results of this section without proof, but the proofs may be found in [GS99].

Definition 3.6. The *Weil algebra* of G is defined as $\mathcal{W} = \bigwedge \mathfrak{g}^* \otimes \operatorname{Sym} \mathfrak{g}^*$, where the elements of $\bigwedge^1 \mathfrak{g}^*$ each have degree 1, and the elements of $\operatorname{Sym}^1 \mathfrak{g}^*$ each have degree 2. The differential $d_{\mathcal{W}}$ on $\mathcal{W}$ is defined on generators as $d_{\mathcal{W}}(x \otimes 1) = 1 \otimes x$ and $d_{\mathcal{W}}(1 \otimes x) = 0$ for $x \in \mathfrak{g}^*$.

The Weil algebra is a particularly nice example of an acyclic G^* algebra of type (C). Setting $E = \mathcal{W}$ in the definition of equivariant cohomology of a G^* algebra, we obtain $H^*_G(M) = H^*((\Omega(M) \otimes \mathcal{W})_{\text{bas}})$.

We now introduce the Cartan complex, which also computes the equivariant cohomology of M, and is especially suitable for computations. This complex is also suitable for extending results from ordinary de Rham cohomology to the equivariant setting, since exact sequences of ordinary differential forms give rise to exact sequences of the corresponding Cartan complexes.

Let $\xi_1, \ldots, \xi_n$ be a basis of $\mathfrak{g}$, and let $\xi_1^*, \ldots, \xi_n^*$ be the corresponding dual basis of $\mathfrak{g}^*$.

Definition 3.7. The *Cartan complex* is defined as $\Omega_G(M) = (\Omega(M) \otimes \operatorname{Sym} \mathfrak{g}^*)^G$, where elements of $\operatorname{Sym}^1 \mathfrak{g}^*$ are assigned degree 2. The differential d_G is defined as

$$d_G(\omega \otimes f) = \sum_{i=1}^{n} \iota_{\xi_i} \omega \otimes (\xi_i^* f).$$

Alternatively, we may think of elements of $\Omega_G(M)$ as G-equivariant polynomial maps $\omega\colon\, \mathfrak{g} \to \Omega(M)$. The differential d_G now becomes $(d_G \omega)(\xi) = d(\omega(\xi)) + \iota_\xi(\omega(\xi))$. From this interpretation it is clear that the definition of $\Omega_G(M)$ is independent of the basis chosen.

4. Properties of equivariant cohomology

In this section we briefly recall some familiar properties of singular cohomology, and state similar properties for equivariant cohomology.

4.1. **Long exact sequences.** Let Y be a topological space with a G-action, such that X is a G-invariant subspace. Then the inclusion $X \hookrightarrow Y$ induces an inclusion $X_G \hookrightarrow Y_G$. Set $H^*_G(Y, X) = H^*(Y_G, X_G)$. The following two results are immediate from the topological definition of equivariant cohomology.

Proposition 4.1. The relative equivariant cohomology fits into the following long exact sequence:

$$\cdots \to H^n_G(Y, X) \to H^n_G(Y) \to H^n_G(X) \to H^{n+1}_G(Y, X) \to \cdots .$$

Proposition 4.2 (Equivariant Mayer-Vietoris sequence)**.** Suppose that $Y = U \cup V$ for two open G-invariant subspaces U and V. Then there is a long exact sequence of equivariant cohomology groups as follows:

$$\cdots \to H^{n-1}_G(U \cap V) \to H^n_G(Y) \to H^n_G(U) \oplus H^n_G(V) \to H^n_G(U \cap V) \to \cdots .$$

Now suppose that Y is a compact smooth manifold such that X is a closed submanifold. Using the Cartan complex, set $H^*_G(Y, X) = H^*(\Omega_G(Y, X))$. Further, let $H^n_G(Y - X)_c$ denote the cohomology of $\Omega_G(Y - X)_c$, which is the Cartan complex with compact support on $X - Y$.

Proposition 4.3. There is a long exact sequence

$$\cdots \to H^n_G(Y - X)_c \to H^n_G(Y) \to H^n_G(X) \to H^{n+1}_G(Y - X)_c \to \cdots .$$

Proof sketch. Using some local computations in an equivariant tubular neighbourhood around X, it can be shown that the extension map $\Omega(Y - X)_c \to \Omega(Y, X)$ induces an isomorphism of H^*_G-modules in cohomology. The result now follows from Proposition 4.1. ∎

4.2. **The push-forward map.** Let X and Y be compact and oriented smooth manifolds, of dimensions m and n respectively. Let $d = n-m$. Suppose that there is a map $f\colon X \to Y$. In this case we can define a push-forward map $f_*\colon H^*(X) \to H^{*+d}(Y)$, as the composition $H^*(X) \cong H_{m-*}(X) \to H_{m-*}(Y) \cong H^{*+d}(Y)$. The first and third maps come from Poincaré duality, while the second is the induced map in homology.

Now suppose that X and Y are each equipped with an orientation-preserving G-action. Let $f\colon X \to Y$ be a G-equivariant map. Since Poincaré duality does not hold for M_G, the previous definition does not immediately extend to equivariant cohomology. Nonetheless, we can construct an equivariant push-forward map. The idea is to first tackle two special cases, and then combine them to construct the general definition.

First let $f\colon X \to Y$ be a fibre bundle. In this case the push-forward in ordinary cohomology is just integration over the fibre. Let $\omega = \sum_i \omega_i \otimes q_i$ be a form in the Cartan complex. The equivariant fibre integral is defined as $f_*\omega = \sum_i (f_*\omega_i) \otimes q_i$, where $f_*\omega_i$ is the fibre integral of ordinary cohomology. This is well-defined as a map from $H_G^*(X)$ to $H_G^*(Y)$.

Next suppose that $f\colon X \hookrightarrow Y$ is the inclusion of a closed submanifold. Let ν_X be the normal bundle of X inside Y. By the tubular neighbourhood theorem, we may assume that ν_X is a subspace of Y. Recall the usual Thom isomorphism $H^{*-d}(X) \cong H^*(\nu_X)_c$. The push-forward in ordinary cohomology is the composition $H^{*-d}(X) \to H^*(\nu_X)_c \to H^*(Y)_c = H^*(Y)$, where the second map is induced from the inclusion $\nu_X \to Y$. For equivariant cohomology, consider an equivariant tubular neighbourhood ν_X of X inside Y. There is an equivariant Thom isomorphism $H_G^{*-d}(X) \to H_G^*(\nu_X)_c$ (see e.g. [GS99]). The equivariant push-forward is defined as the composition $H_G^{*-d}(X) \to H_G^*(\nu_X)_c \to H_G^*(Y)_c = H_G^*(Y)$.

Finally we define the equivariant push-forward for any $f\colon X \to Y$. We can write f as the composition $X \xrightarrow{\Gamma} X \times Y \xrightarrow{\pi} Y$, where Γ is the graph map and π is the second projection. Since Γ realizes X as a submanifold of $X \times Y$, the map $\Gamma_*\colon H_G^*(X) \to H_G^{*+n}(X \times Y)$ is defined. Since $\pi\colon X \times Y \to Y$ is a fibre bundle, the map $\pi_*\colon H_G^*(X \times Y) \to H_G^{*-m}(Y)$ is also defined. Now set $f_*\colon H_G^*(X) \to H_G^{*+d}(Y)$ to be the composition $\pi_* \circ \Gamma_*$.

With these definitions, the equivariant push-forward is well-defined and satisfies properties analogous to the usual push-forward. We mention a few in particular, which will be important later. First, if there are maps $f\colon X \to Y$ and $g\colon Y \to Z$, then $(fg)_* = f_* g_*$. Also, if $\omega \in H_G^*(X)$ and $\eta \in H_G^*(Y)$, then $f_*(\omega \cdot f^*\eta) = (f_*\omega) \cdot \eta$. If $E \to X$ is a G-equivariant bundle, then we can define its equivariant Euler class (denoted $\mathrm{Eu}_G(E)$). Then if $f\colon X \hookrightarrow Y$ is the inclusion of a submanifold and ν_X is the equivariant normal bundle to X, then $f^* f_* \omega = \mathrm{Eu}_G(\nu_X)\omega$. More details may be found in [GS99].

5. Reduction to a maximal torus

The definitions of equivariant cohomology discussed so far are valid for any compact connected Lie group. However it is more convenient for computational purposes to consider compact connected abelian Lie groups, which are compact tori (products of S^1). Since G contains a maximal torus T, we can restrict the G-action on M to a T-action on M. In this section, we show that we can in fact compute $H_G^*(M)$ in terms of $H_T^*(M)$ and the action of the Weyl group.

Let r be the rank of G. Fix a maximal torus T inside G. Set $\mathfrak{t} = (\mathrm{Lie}\, T)_{\mathbb{C}}$. We first prove the following proposition.

Proposition 5.1. There is a natural isomorphism $H_T^* \cong \mathrm{Sym}\, \mathfrak{t}^*$.

Proof. Recall that the classifying space of T is an r-fold product of the infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$. The cohomology of this space is a polynomial algebra on n generators, each of degree 2.

Consider the fibre bundle $T \to ET \to BT$. We compute the E_2 page of the Serre spectral sequence for this fibre bundle. Since BT is simply connected, we obtain $E_2^{p,q} = H^p(BT) \otimes H^q(T)$, with maps $d_2^{p,q}\colon E_2^{p,q} \to E_2^{p+2,q-1}$. Since ET is contractible, it has no cohomology except in degree 0. Hence we obtain a natural isomorphism $E_2^{0,1} \xrightarrow{\cong} E_2^{2,0}$. We can rewrite this isomorphism more explicitly as follows:

$$H^1(T) \cong H^0(BT) \otimes H^1(T) \xrightarrow{d_2^{0,1}} H^2(BT) \otimes H^0(T) \cong H^2(BT).$$

It is known that $H^*(T) = (\bigwedge \mathfrak{t}^*)^T = \bigwedge \mathfrak{t}^*$. Hence $H^1(T) \cong \mathfrak{t}^*$. Therefore we can naturally identify $H^2(BT)$ with $\mathfrak{t}^*$. Since $H^*(BT) = \mathrm{Sym}(H^2(BT))$, it follows that $H^*(BT) \cong \mathrm{Sym}\, \mathfrak{t}^*$. ∎

Now we can prove that for a general compact connected Lie group G with maximal torus T, we can essentially reduce the computations of G-equivariant cohomology to computations of T-equivariant cohomology. Let N be the normalizer of T in G. Let $W = N/T$ be the Weyl group.

Theorem 5.2. *There is a natural isomorphism $H_G^*(M) \cong H_T^*(M)^W$.*

Proof. As before, let M_G and M_T denote the homotopy quotients of M by G and T respectively. The action of N on $EG \times M$ (by restriction of the G-action) descends to M_T. Moreover, the subgroup $T \subset N$ acts trivially on M_T. Hence we obtain a right action of W on M_T, which induces a left action of W on $H_T^*(M)$. There is a quotient map $f\colon M_T \to M_G$ which induces a map $f^*\colon H_G^*(M) \to H_T^*(M)$. To prove the theorem, it suffices to show that f^* is injective, and that the image of f^* is $H_T^*(M)^W$.

We can factor f as the composition $M_T \to M_N \to M_G$. The first map $M_T \to M_N$ is a fibre bundle with fibre W. Since W is finite, the induced map on cohomology gives an isomorphism $H^*(M_N) \cong H^*(M_T)^W$.

The second map $M_N \to M_G$ is a fibre bundle with fibre G/N. First we show that G/N has trivial cohomology. Using the fibering $G/T \to G/N$ with fibre W, see that $H^*(G/N) \cong H^*(G/T)^W$. It is known (see e.g. [GHV76]) that as a W-module, $H^*(G/T)$ is isomorphic to the regular representation. Hence $H^*(G/N) \cong H^*(G/T)^W \cong \mathbb{C}$. Finally, using the Serre spectral sequence for the fibre bundle $M_N \to M_G$, we deduce that $H^*(M_G) \cong H^*(M_N)$.

Combining the two pieces, we see that $H^*(M_G) \cong H^*(M_N) \cong H^*(M_T)^W$, which proves the theorem. ☐

6. Localization theorems

In this section, we assume that M is a compact and oriented smooth manifold, such that the G-action is smooth and orientation-preserving. As outlined in Section 5, we may express $H_G^*(M)$ as the Weyl group invariants of $H_T^*(M)$. Therefore in this section, we work with a compact torus T. Let M^T denote the T-fixed subset of M.

The main localization theorem states that almost all of the equivariant cohomology of M is governed by the equivariant cohomology of M^T. To make this notion precise, we use the H_T^*-module structure on the cohomology ring $H_T^*(M)$. Recall from Proposition 5.1 that $H_T^* \cong \operatorname{Sym} \mathfrak{t}^* \cong \mathbb{C}[\mathfrak{t}]$. Hence the support of any H_T^*-module A (denoted $\operatorname{Supp} A$) may be identified as a subset of $\mathfrak{t}$.

Theorem 6.1 (Atiyah-Bott localization)**.** *The kernel and cokernel of the map $i^*\colon H_T^*(M) \to H_T^*(M^T)$ are torsion modules over H_T^*.*

The localization theorem allows us to express the integral of an equivariant form over M in terms of the integrals of the form over the connected components of M^T. Let $M^T = \coprod F$ be the decomposition of M^T into its connected components. Precisely, the integration formula states the following.

Theorem 6.2. *The integral $\int_M \omega$ of some $\omega \in H_T^*(M)$ can be computed as follows:*

$$\int_M \omega = \sum_F \int_F \left(\frac{i_F^* \omega}{\mathrm{Eu}_T(\nu_F)} \right).$$

In the remainder of this section, we first sketch the proofs of these theorems. Then we discuss the Berline-Vergne localization theorem and the Berline-Vergne integration formula.

Lemma 6.3. Let K be a closed subgroup of T. Suppose that for some manifold X, there is a T-equivariant map $X \to T/K$. Then $\operatorname{Supp} H_T^*(X)$ is contained in $(\operatorname{Lie} K)_{\mathbb{C}}$.

Proof. The support of H_K^* as an H_T^*-module is equal to $(\operatorname{Lie} K)_{\mathbb{C}}$. Consider the map $\pi\colon X \to \{p\}$ that collapses all points of X to p. The induced map $\pi^*\colon H_T^* \to H_T^*(X)$ makes $H_T^*(X)$ into an H_T^*-algebra. To show that the support of $H_T^*(X)$ is contained in $(\operatorname{Lie} K)_{\mathbb{C}}$, it is enough to show that the map π^* factors through H_K^* as a map of H_T^*-algebras.

The map $\pi\colon X \to \{p\}$ may be factored as the composition $X \to T/K \to \{p\}$. In cohomology, we obtain $H_T^* \to H_T^*(T/K) \to H_T^*(X)$. It is easy to check that $H_T^*(T/K) \cong H_K^*$ as H_T^*-algebras. Hence we obtain a factoring $H_T^* \to H_K^* \to H_T^*(X)$. ☐

Let M^T denote the subset of M consisting of points fixed by T. For any point $p \in M$, let T_p denote the isotropy group of p inside T. By compactness of M and the equivariant tubular neighbourhood theorem, we can show that the set $\{T_p \mid p \in M\}$ is finite. We now prove the following proposition, which shows that away from M^T, the T-equivariant cohomology is a torsion module over H_T^*.

Proposition 6.4. The supports of the H_T^*-modules $H_T^*(M - M^T)$ and $H_T^*(M - M^T)_c$ lie in the finite union $\bigcup_p(\mathrm{Lie}\, T_p)_{\mathbb{C}}$, where $p \in M - M^T$. In particular, both are torsion modules over H_T^*.

Proof sketch. Let $p \in M - M^T$. Then T_p is a proper closed subgroup of T. Since T is compact, we can identify T/T_p with the T-orbit of p, which makes T/T_p a smooth submanifold of M. Let U_p be a T-invariant tubular neighbourhood of T/T_p in M. Then there is a T-equivariant projection map $U_p \to T/T_p$. Using Lemma 6.3, we see that $H_T^*(U_p)$ is supported in $(\mathrm{Lie}\, T_p)_{\mathbb{C}}$.

Now let U be a T-invariant tubular neighbourhood of M^T. Since $M - U$ is compact, there is a finite open cover $\mathcal{U} = \{U_p\}$ of $M - U$, where U_p is as above. Recall that for a short exact sequence of modules $0 \to A \to B \to C \to 0$ over some ring R, we have $\mathrm{Supp}(B) \subset \mathrm{Supp}(A) \cup \mathrm{Supp}(C)$. Using this fact and the equivariant Mayer-Vietoris sequence finitely many times for the open cover $\mathcal{U}$, we see that

$$\mathrm{Supp}(H_T^*(M - U)) \subset \bigcup_{U_p \in \mathcal{U}} \mathrm{Supp}(H_T^*(U_p)) \subset \bigcup_{p \in M - M^T} (\mathrm{Lie}\, T_p)_{\mathbb{C}}.$$

Since there is a T-equivariant deformation retract of U onto M^T, it is clear that $H_T^*(M-U) \cong H_T^*(M-M^T)$ as H_T^*-modules. Hence the proposition is proved for $H_T^*(M - M^T)$.

Using the Cartan complex, it is easy to see that $H_T^*(M - M^T)_c$ is a module over $H_T^*(M - M^T)$. Also, the action of elements of H_T^* on $H_T^*(M-M^T)_c$ factors through the ring map $H_T^* \to H_T^*(M-M^T)$. Therefore the support of $H_T^*(M - M^T)_c$ as an H_T^*-module is contained in the support of $H_T^*(M - M^T)$ as an H_T^*-module. Hence the proposition is proved for $H_T^*(M - M^T)_c$ as well. □

Now we can finish the proof of the Atiyah-Bott localization theorem.

Proof. Recall from Proposition 4.3 that there is a long exact sequence of cohomology groups as follows:

$$\cdots \to H_G^n(M - M^T)_c \to H_G^n(M) \xrightarrow{i^*} H_G^n(M^T) \to H^{n+1}(M - M^T)_c \to \cdots.$$

The result is now clear from Proposition 6.4. □

By the localization theorem, the restriction i^* is invertible modulo torsion. The following proposition allows us to produce an explicit inverse.

Proposition 6.5. The kernel and cokernel of the push-forward map $i_*\colon H_T^*(M^T) \to H_T^*(M)$ are torsion modules over H_T^*.

Proof. Since i_* factors through the equivariant Thom isomorphism, the kernel and cokernel of i_* correspond to the kernel and cokernel of the map $H_T^*(\nu_{M^T})_c \to H_T^*(M)_c$, which are precisely $H_T^{*-1}(M - \nu_{M^T})_c$ and $H_T^*(M - \nu_{M^T})_c$. These are torsion modules by Proposition 6.4. □

Recall that the composition $i^* i_*$ is multiplication by $\mathrm{Eu}_T(\nu_{M^T})$, which is the equivariant Euler class of the normal bundle of M^T. Since both i^* and i_* are isomorphisms modulo torsion, so is their composition. In particular, $\mathrm{Eu}_T(\nu_{M^T})$ is invertible over some dense open set of $\mathfrak{t}$, and the maps i_* and $\mathrm{Eu}_T^{-1}(\nu_{M^T})i^*$ are mutually inverse over this set. Rewriting in terms of the connected components of M^T, we see that the inverse of $i_* = \sum_F i_*^F$ is $\sum_F \mathrm{Eu}_T^{-1}(\nu_F) i_F^*$. Now we can finish the proof of the integration formula.

Proof of the integration formula. We first work over an open dense subset of $\mathfrak{t}$ where $\mathrm{Eu}_T^{-1}(\nu_F)$ is defined for every F. Let $\omega \in H_T^*(M)$. Let $\pi\colon M \to \{p\}$ be the map that collapses all points of M to p. Then we can compute $\int_M \omega$ as follows:

$$\int_M \omega = \pi_* \omega = \sum_F \pi_* i_*^F \left(\frac{i_F^* \omega}{\mathrm{Eu}_T(\nu_F)} \right) = \sum_F \int_F \left(\frac{i_F^* \omega}{\mathrm{Eu}_T(\nu_F)} \right).$$

However, the left hand side of the equality is an element of H_T^* (hence a polynomial function on $\mathfrak{t}$), while the right hand side is a sum of rational functions. Since this identity holds on an open dense subset of $\mathfrak{t}$, it holds as a formal identity of rational functions. □

We now discuss the Berline-Vergne localization theorem. This theorem is stated in terms of a modified version of the Cartan complex, and the proof is by an explicit computation using differential forms. Fix some $\xi \in \mathfrak{t}$. Consider the complex $\Omega_\xi(M) = \{\omega \in \Omega(M) \mid L_\xi \omega = 0\}$. The differential d_ξ on $\Omega_\xi(M)$ is defined as $d_\xi \omega = d\omega + i_\xi \omega$, where d is the de Rham differential on $\Omega(M)$. Let v_ξ be the vector field on M generated by ξ, as defined in (1). Let M^ξ denote the submanifold of M consisting of the zeroes of v_ξ. It is clear that $d_\xi = d$ on M^ξ, which means that $H^*(\Omega_\xi(M^\xi)) = H^*(\Omega(M^\xi)) = H^*(M^\xi)$. Let $i\colon M^\xi \hookrightarrow M$ be the inclusion. In this situation, we have the following version of the localization theorem.

Theorem 6.6 (Berline-Vergne localization)**.** *The map $i^*\colon H^*(\Omega_\xi(M)) \to H^*(M^\xi)$ is an isomorphism.*

Remark 6.7. Let ξ be sufficiently generic, so that $M^\xi = M^T$. Now observe that the complex $\Omega_\xi(M)$ is the evaluation of the Cartan complex at $\xi \in \mathfrak{t}$. Therefore in this case, the map $i^*\colon H^*(\Omega_\xi(M)) \to H^*(M^\xi)$ corresponds to restricting the H_T^*-module map $H_T^*(M) \to H_T^*(M^T)$ to the corresponding fibres over the point ξ. Hence Theorem 6.6 may be thought of as a fibre-wise version of Theorem 6.1.

To prove Theorem 6.6, we first need the following lemma, which closely resembles Proposition 6.4.

Lemma 6.8. If N is a T-invariant submanifold of M that does not intersect M^ξ, then $H^*(\Omega_\xi(N)) = 0$.

Proof. Fix a T-invariant Riemannian metric g on M. Construct a 1-form α on N, defined on a vector field Y as follows:

$$\alpha(Y) = \frac{g(v_\xi, Y)}{g(v_\xi, v_\xi)}.$$

It can be checked that $\iota_\xi \alpha = 1$ and $L_\xi \alpha = 0$. A computation shows that $d_\xi(1 + d\alpha) = 0$. Finally, $1 + d\alpha$ is invertible in $\Omega_\xi(N)$, since $d\alpha$ is nilpotent. Now if ω is a d_ξ-cocycle in $\Omega_\xi(N)$, we can present ω as a d_ξ-coboundary as follows: $\omega = d_\xi(\alpha(1 + d\alpha)^{-1}\omega)$. □

Proof of Theorem 6.6. The first step of the proof is to show that for any T-invariant open neighbourhood U of M^ξ, the map $H^*(\Omega_\xi(M)) \to H^*(\Omega_\xi(U))$ is an isomorphism. The second step is to show that for any T-equivariant tubular neighbourhood U of M^ξ, the map $H^*(\Omega_\xi(U)) \to H^*(M^\xi)$ is an isomorphism.

For the first step, we show that $H^*(\Omega_\xi(M)) \to H^*(\Omega_\xi(U))$ is both surjective and injective. Let ρ be a function on M that is compactly supported on U, and which is identically equal to 1 on a neighbourhood of M^ξ. Construct a 1-form $\alpha \in \Omega_\xi(M - M^\xi)$ as in the proof of the previous lemma.

To show surjectivity, let ω be a d_ξ-cocycle in $\Omega_\xi(U)$. Consider the form $\omega' = \omega - d_\xi((1-\rho)\alpha(1 + d\alpha)\omega)$, which represents the same class in cohomology. The form ω' is compactly supported in U. Hence the class of ω' in $H^*(\Omega_\xi(M))$ restricts to the class of ω in $H^*(\Omega_\xi(U))$. To show injectivity, let ω be a d_ξ-cocycle in $\Omega_\xi(M)$ such that $\omega = d_\xi \eta$ on U. Then $\omega' = \omega - d_\xi(\rho\eta)$ represents the same cohomology class, but is supported in $M - U$. By Lemma 6.8, ω' must be zero in cohomology.

For the second step, suppose that U is a T-equivariant tubular neighbourhood of M^ξ. Then there are T-equivariant maps $i\colon M^\xi \to U$ (inclusion) and $\pi\colon U \to M^\xi$ (projection). The composition $\pi \circ i$ is the identity map, while the composition $i \circ \pi$ is homotopic to the identity map by a T-equivariant homotopy. It is possible to construct a chain homotopy $h\colon \Omega^n(U) \to \Omega^{n+1}(U)$ in the usual way, with the additional property that $\iota_\xi h = -h\iota_\xi$.

It is clear that $i^*\colon H^*(\Omega_\xi(U)) \to H^*(M^\xi)$ is surjective, since $i^*\pi^*$ is the identity map. To show injectivity, let $\omega \in \Omega_\xi(U)$ be a d_ξ-cochain such that $i^*\omega = 0$. Then a computation shows that $d_\xi(h\omega) = \omega$, which means that ω is a coboundary. □

As before, this version of the localization theorem can also be used to deduce an integration formula. In this case, the proof involves an explicit computation using differential forms and Stokes' theorem. We only discuss the case in which the vector field v_ξ has isolated zeroes. The general case is more complicated, but is proved in a similar way.

Theorem 6.9. *Let $\xi \in \mathfrak{t}$ such that v_ξ has isolated zeroes $\{p_i\}$ on M. Then for any cohomology class $\omega \in H^*(\Omega_\xi(M))$, we can compute its integral over M as follows:*

$$\int_M \omega = \sum_i \frac{\omega(p_i)}{\mathrm{Eu}_T(\nu_{p_i})}.$$

Proof sketch. Let ω be a d_ξ-cocycle in $\Omega_\xi(M)$. As in the proof of Theorem 6.6, construct a form α on $M - M^\xi$ with the property that $\iota_\xi \alpha = 1$ and $L_\xi \alpha = 0$. On $M - M^\xi$, we can write $\omega = d_\xi(\alpha(1+d\alpha)^{-1}\omega)$.

Consider a small open ball U_i^ϵ of radius ϵ around each fixed point p_i. Let $V^\epsilon = \bigcup_i U_i^\epsilon$. By Stokes' theorem, we have

$$\int_M \omega = \lim_{\epsilon \to 0} \int_{M - V^\epsilon} \omega = -\lim_{\epsilon \to 0} \int_{\partial V^\epsilon} \alpha(1+d\alpha)^{-1}\omega = -\lim_{\epsilon \to 0} \sum_i \int_{\partial U_i^\epsilon} \alpha(1+d\alpha)^{-1}\omega.$$

Consider the map $\varphi\colon \partial V^\epsilon \to M^\xi$ such that $\varphi(\partial U_i^\epsilon) = p_i$. It is possible to construct forms β_i in local coordinates on $U_i^{2\epsilon} - \{p_i\}$ with the property that $\varphi_*(\beta_i(1+d\beta_i)^{-1}) = -\operatorname{Eu}_T^{-1}(\nu_{p_i})$, and such that $\iota_\xi \beta_i = 1$ and $L_\xi \beta_i = 0$. In this case for every i, we have

$$\int_{\partial U_i^\epsilon} \alpha(1+d\alpha)^{-1}\omega = \int_{\partial U_i^\epsilon} \beta_i(1+d\beta_i)^{-1}\omega.$$

Recall from Lemma 6.8 that ω is cohomologous to $\varphi^*(\omega(p_i))$ on U_i^ϵ. Using this fact, we see that

$$\int_M \omega = -\lim_{\epsilon \to 0} \sum_i \int_{\partial U_i^\epsilon} \beta_i(1+d\beta_i)^{-1}\varphi^*(\omega(p_i)) = \lim_{\epsilon \to 0} \sum_i \frac{\omega(p_i)}{\operatorname{Eu}_T(\nu_{p_i})} = \sum_i \frac{\omega(p_i)}{\operatorname{Eu}_T(\nu_{p_i})}.$$

□

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