Bernstein–Sato polynomials and monodromy conjectures for Weyl arrangements

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Introduction to $\mathscr{D}$-modules and Bernstein–Sato polynomials

What is a $\mathscr{D}$-module?

Differential operators on a space form a ring $\mathscr{D}$.

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Example

On $\mathbb{C}^2$, the ring $\mathscr{D}$ is generated by ∂_x, ∂_y, and polynomials in x and y.

Some examples of differential operators:

$$\partial_x\partial_y, \quad x\partial_y + y, \quad (x^2 + y)\partial_x^2\partial_y + y\partial_y.$$

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Example (on $\mathbb{C}^2$)

$\mathscr{D}$ acts on the space of polynomials $\mathbb{C}[x, y]$. For example:

$$\left(y\partial_x + x\right) \cdot x^2 = 2yx + x^3.$$

So $\mathbb{C}[x, y]$ is a $\mathscr{D}$-module.

What is a $\mathscr{D}$-module?

A $\mathscr{D}$-module is a left module over $\mathscr{D}$.

Example

$\mathscr{D}f^{-1}$ is the $\mathscr{D}$-module generated by $(1/f)$. Elements:

$$\partial_x f^{-1},\ y\partial_y f^{-1},\ x^2 f^{-1},\ \text{etc.}$$

An interesting invariant of $\mathscr{D}$-modules

The Bernstein–Sato polynomial (or the *b*-function) is an invariant attached to a $\mathscr{D}$-module.

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Case of interest

The b-function of $\mathscr{D}f^{-1}$, also called the b-function of f.

What is the b-function of f?

Theorem (Bernstein)
For any polynomial f, there is some differential operator L and some polynomial $b(n)$ such that

$$L \cdot (f^{n+1}) = b(n) f^n.$$

The minimal monic polynomial $b(n)$ satisfying such an equation is called the b-function of f.

What is the b-function of f?

Definition/Theorem

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Example: $f(x) = x$

$$\partial_x \cdot (x^{n+1}) = (n+1)\, x^n.$$

$$b(n) = (n+1).$$

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Example: $f(x,y) = xy$

$$\partial_x \partial_y \cdot (xy)^{n+1} = (n+1)^2 (xy)^n.$$

$$b(n) = (n+1)^2.$$

What is the b-function of f?

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Minimal monic polynomial $b(n)$ such that $L\cdot(f^{n+1}) = b(n)f^n$.

Example: $f(x,y) = x^3 + y^2$

$$\frac{1}{216}(18x\partial_x\partial_y^2+8\partial_x^3+54n\partial_y^2+81\partial_y^2)\cdot(x^3+y^2)^{n+1} = (n+1)\left(n+\frac{5}{6}\right)\left(n+\frac{7}{6}\right)(x^3+y^2)^n.$$

$$b(n) = (n+1)\left(n+\frac{5}{6}\right)\left(n+\frac{7}{6}\right).$$

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Minimal monic polynomial $b(n)$ such that $L \cdot (f^{n+1}) = b(n) f^n$.

Example: $f(x, y) = x^3 + y^4$

$$
\begin{aligned}
L = {} & 248832 y^2 \partial_x^3 \partial_y^2 n^2 + 497664 y^2 \partial_x^3 \partial_y^2 n + 245952 y^2 \partial_x^3 \partial_y^2 - 104976 y \partial_y^5 n^2 \\
& - 209952 y \partial_y^5 n - 103761 y \partial_y^5 + 663552 y \partial_x^3 \partial_y n^3 + 3234816 y \partial_x^3 \partial_y n^2 + 4460544 y \partial_x^3 \partial_y n \\
& + 1874880 y \partial_x^3 \partial_y + 559872 \partial_y^4 n^3 + 1469664 \partial_y^4 n^2 + 1257768 \partial_y^4 n + 350406 \partial_y^4 \\
& + 1327104 \partial_x^3 n^4 + 6635520 \partial_x^3 n^3 + 12699648 \partial_x^3 n^2 + 10764288 \partial_x^3 n + 3363136 \partial_x^3.
\end{aligned}
$$

$$
b(n) = (n+1)\left(n + \frac{5}{6}\right)\left(n + \frac{7}{6}\right)\left(n + \frac{7}{12}\right)\left(n + \frac{11}{12}\right)\left(n + \frac{13}{12}\right)\left(n + \frac{17}{12}\right).
$$

Singularity invariants and the monodromy conjecture

The b-function and geometry

Question

What does the b-function tell us about the geometry of $V(f)$?

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Remarks

- $V(f)$ is smooth if and only if $b(n) = (n+1)$.

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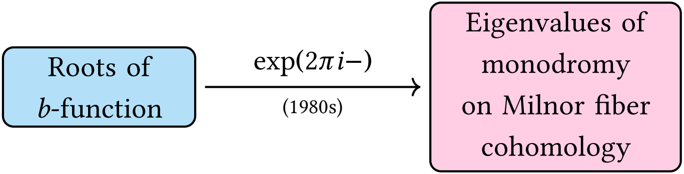
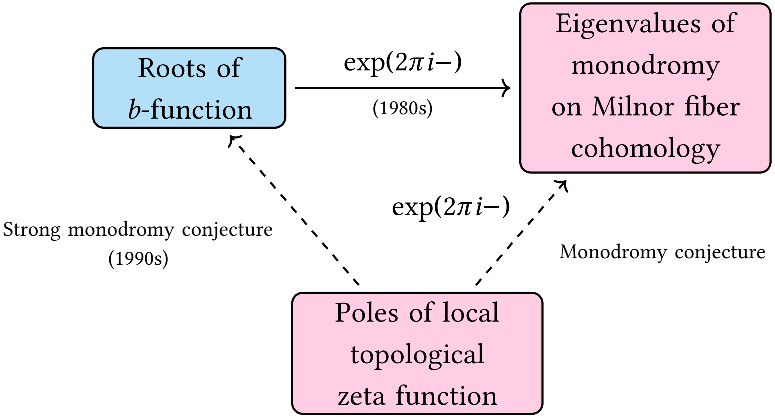
Remarks

- $V(f)$ is smooth if and only if $b(n) = (n+1)$.
- The largest root of $b(n)$ is the negative of the log canonical threshold of f.

The b-function and geometry



The b-function and geometry



Weyl hyperplane arrangements

Hyperplane arrangements formed by the root systems of semisimple Lie algebras.

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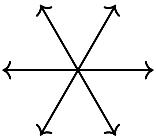
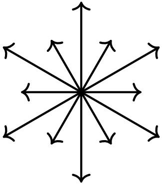
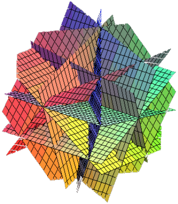
Examples: A2, G2, and B3 arrangements







(Source: John Stembridge)

Main theorem

Theorem (B.–Walters 2015)
The strong monodromy conjecture holds for all Weyl hyperplane arrangements.

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That is, every pole of the LTZF is a root of the *b*-function.

Proof sketch of main theorem

Observations

- (Budur–Mustaţă–Teitler 2011) It is sufficient to check that one particular pole of the LTZF is a root of the *b*-function.

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Proof sketch of main theorem

Observations

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Key lemma

The b-function of the second polynomial divides the b-function of the first polynomial.

Some computations and further directions

The b-function of the Vandermonde determinant

The type A_n Weyl arrangement is cut out by the following polynomial:

$$V_n = \prod_{1 \le i < j \le n} (x_j - x_i) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$$

This polynomial is called the Vandermonde determinant.

The b-function of the Vandermonde determinant

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$$\begin{aligned} b_{V_1}(s) &= 1 \qquad \text{(by convention)} \\ b_{V_2}(s) &= (s+1) \\ b_{V_3}(s) &= (s+1)\cdot\left(s+\frac{2}{3}\right)\left(s+\frac{3}{3}\right)\left(s+\frac{4}{3}\right) \\ b_{V_4}(s) &= b_{V_3}(s)\cdot\left(s+\frac{3}{6}\right)\left(s+\frac{4}{6}\right)\cdots\left(s+\frac{9}{6}\right) \end{aligned}$$

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b_{V_4}(s) &= b_{V_3}(s)\cdot\left(s+\frac{3}{6}\right)\left(s+\frac{4}{6}\right)\cdots\left(s+\frac{9}{6}\right) \\
b_{V_5}(s) &= b_{V_4}(s)\cdot\left(s+\frac{4}{10}\right)\left(s+\frac{5}{10}\right)\cdots\left(s+\frac{16}{10}\right)
\end{aligned}
$$

Theorem (B.–Walters 2015)

We have a divisibility relation as follows:

$$b_{V_n}(s) \,\Big|\, c_n(s) \cdot \prod_{i=(n-1)}^{(n-1)^2} \left(s + \frac{i}{\binom{n}{2}} \right).$$

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Here, $c_n(s)$ is a recursive expression in terms of the b-functions of smaller Vandermonde determinants.

Conjecture

The divisibility relation in the previous theorem is an equality.

Further directions

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Namely,

$$b_{V_n}(s) = c_n(s) \cdot \prod_{i=(n-1)}^{(n-1)^2} \left(s + \frac{i}{\binom{n}{2}} \right).$$

Questions

- Can we compute the b-functions of all Weyl arrangements?
- What about other natural symmetric polynomials arising from Lie theory?

Thank you!