Cohomology of perverse sheaves on T-varieties

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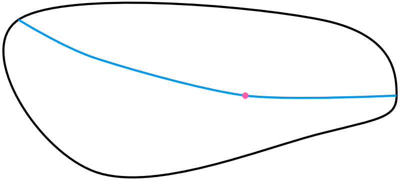
What is a perverse sheaf?

Let X be a complex algebraic variety with a fixed stratification.



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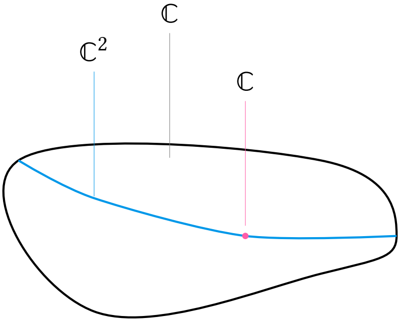
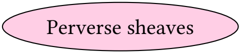
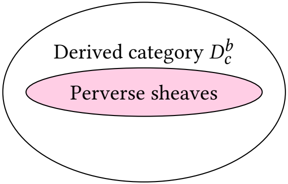


Illustration of a perverse sheaf on X

Cohomology of perverse sheaves



Cohomology of perverse sheaves



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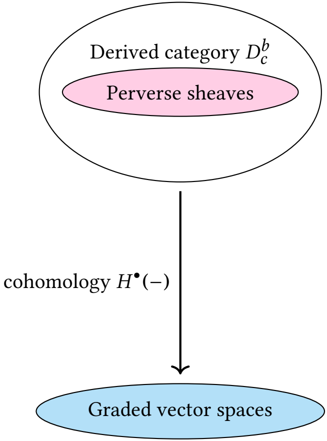

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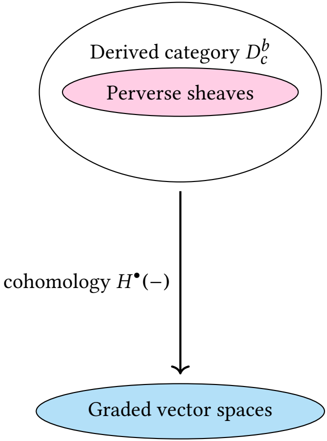

Natural operations,
e.g. $\oplus$, $\underline{\mathrm{RHom}}$, $\otimes$, $\mathbb{D}$

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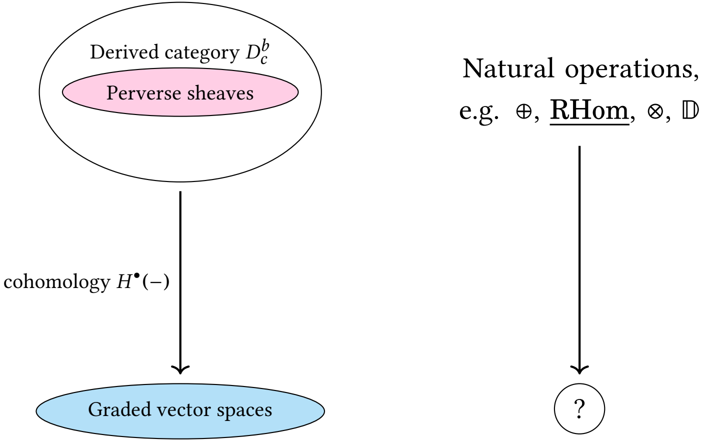

Some extra structure

If $\underline{X}$ is the constant sheaf on X, then $C = H^\bullet(\underline{X})$ is just the singular cohomology of X.

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If $\mathscr{F}$ is any object in $D^b_c(X)$, then $H^\bullet(\mathscr{F})$ has a left and right C-action:

$$C \otimes H^\bullet(\mathscr{F}) \to H^\bullet(\underline{X} \otimes \mathscr{F}) \cong H^\bullet(\mathscr{F}),$$
$$H^\bullet(\mathscr{F}) \otimes C \to H^\bullet(\mathscr{F} \otimes \underline{X}) \cong H^\bullet(\mathscr{F}).$$

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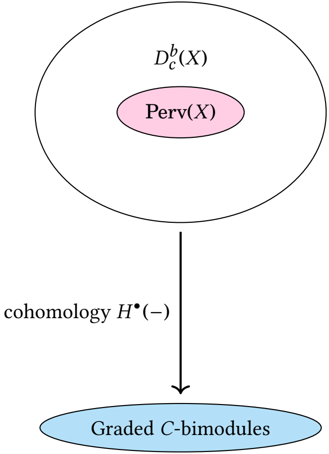
So $H^\bullet$ is a functor from $D_c^b(X)$ to graded C-bimodules.

Cohomology of perverse sheaves



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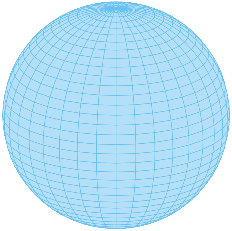
Decomposition of a $\mathbb{C}^*$-variety

$N = \infty$



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$\mathbb{C}P^1$

Consider a variety with a $\mathbb{C}^*$-action, and decompose it into *attracting Białynicki-Birula cells.*

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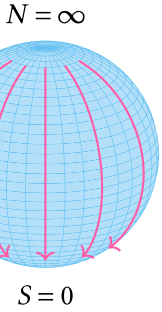

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For example:

$$X_0 = \{x \in X \mid \lim_{t \to 0} t \cdot x = 0\} = \mathbb{C}$$

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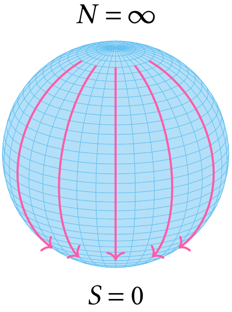

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We study perverse sheaves adapted to such a stratification.

Setup

Let X be a smooth projective T-variety with finitely many fixed points. Fix a one-parameter subgroup $\lambda\colon \mathbb{C}^* \to T$ such that $X^\lambda = X^T$.

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Example
Flag varieties G/P for a reductive algebraic group G.

Observations

By functoriality of $H^\bullet$, there is always a natural map

$$\mathrm{Hom}^i_D(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}^i_C(H^\bullet(\mathscr{F}), H^\bullet(\mathscr{G})).$$

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Similarly, restriction to the diagonal gives a natural "multiplication" map

$$H^\bullet(\mathscr{F}_1) \otimes_C \cdots \otimes_C H^\bullet(\mathscr{F}_n) \to H^\bullet(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n).$$

Some known results for $\underline{\mathrm{RHom}}$

Theorem (Ginzburg)
Let X be a T-variety as before. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are simple perverse sheaves on X, then

$$\mathrm{Hom}^i_D(\mathscr{F}_1, \mathscr{F}_2) \cong \mathrm{Hom}^i_C(H^\bullet(\mathscr{F}_1), H^\bullet(\mathscr{F}_2)).$$

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Theorem (Achar–Rider)
The same result as above for parity sheaves adapted to the Bruhat stratification on a generalized Kac-Moody flag variety.

Theorem (B.)
Let X be a T-variety as before. The multiplication map on the (T-equivariant) cohomology of simple perverse sheaves is an isomorphism:

$$H^\bullet(\mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_n) \cong H^\bullet(\mathscr{F}_1) \otimes_C \cdots \otimes_C H^\bullet(\mathscr{F}_n).$$

Proof sketch

We use upward induction on closures of the strata: on the 0-dimensional piece, the isomorphism is easy to check.

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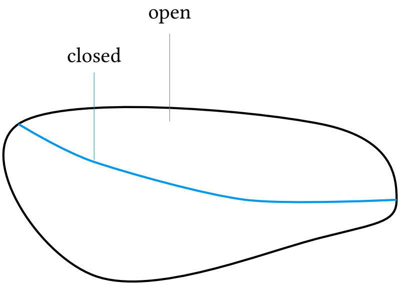
The closure of a larger stratum can be split up into the (open) stratum and the (closed) union of lower strata.



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Since the two stratifications have transverse intersections, this is computable. □

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Proof.
This is achieved by comparing weights (of mixed Hodge modules). □

Questions

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- What are the consequences in representation theory?

Thank you!