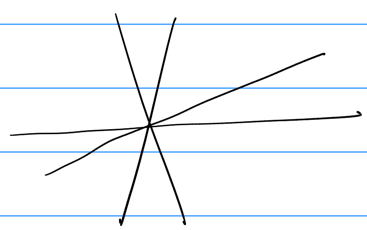
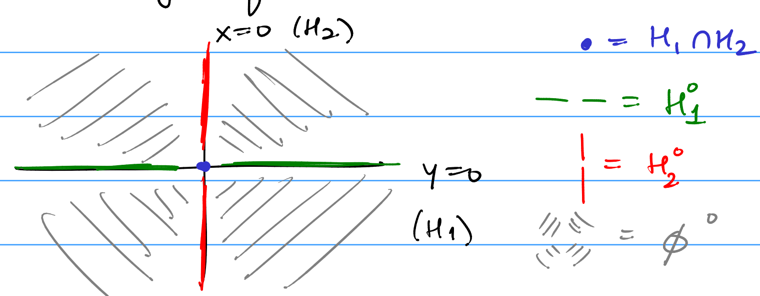
Melbourne pure maths seminar talk 25 May 2018

Title: Perverse sheaves on hyperplane arrangements and gluing

Outline: Work in the setting of arrangements of hyperplanes in $\mathbb{C}^n$. This has lots of combinatorics & also topology — encoded by an appropriate category of "perverse sheaves". Notoriously technical, so the aim is to translate various constructions about them into algebra.

Setting: A hyperplane arrangement $\mathcal{H}$ in $\mathbb{C}^n$. We assume that all hyperplanes of $\mathcal{H}$ have defining equations over $\mathbb{R}$, and let $\mathcal{H}_{\mathbb{R}} \subseteq \mathbb{R}^n$ be the restriction to $\mathbb{R}^n$.
For each $H \in \mathcal{H}_{\mathbb{R}}$, fix a defining equation $f_H = 0$.





Some combinatorics:
For each $x \in \mathbb{R}^n$ & $H \in \mathcal{H}_{\mathbb{R}}$, we have the "real sign" of x w.r.t. H, defined as:

$$\sigma_{\mathbb{R}}(x)_H := \begin{cases} + & \text{if } f_H(x) > 0 \\ - & \text{if } f_H(x) < 0 \\ 0 & \text{if } f_H(x) = 0 \end{cases}$$

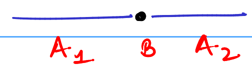
Combine these into the real sign vector $\sigma_{\mathbb{R}}(x) := (\sigma_{\mathbb{R}}(x)_H)_{H \in \mathcal{H}_{\mathbb{R}}}$

For each $z \in \mathbb{C}^n$ & $H \in \mathcal{H}$, we have the "complex sign" of z w.r.t H, defined as:

$$\sigma(z)_H := \begin{cases} * & \text{if } f_H(z) \neq 0 \\ 0 & \text{if } f_H(z) = 0 \end{cases}$$

Combine these to form the complex sign vector $\sigma(z) := (\sigma(z)_H)_{H \in \mathcal{H}}$

Let $\mathcal{C}$ = real face poset = equivalence classes of $\mathbb{R}^n$ under $\sigma_{\mathbb{R}}$.
If A & $B \in \mathcal{C}$, then we say $B \leq A$ if $B \subseteq \bar{A}$ in $\mathbb{R}^n$.

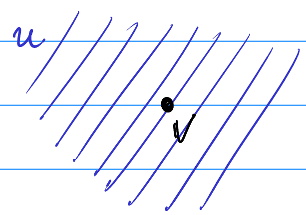


$B \leq A_1$; $B \leq A_2$

$$\begin{array}{ccc} A_1 & & A_2 \\ & \nwarrow \;\; \nearrow & \\ & B & \end{array}$$

Let $\mathcal{S}$ = complex stratification = equivalence classes of $\mathbb{C}^n$ under σ.
If $\mathcal{U}, V \in \mathcal{S}$, we say $V \leq \mathcal{U}$ if $V \subseteq \bar{\mathcal{U}}$ in $\mathbb{C}^n$.

$[\mathbb{C}^1]$:



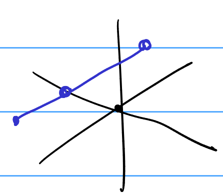
$V \leq \mathcal{U}$

$$\begin{array}{c} \mathcal{U} \\ | \\ V \end{array}$$

→ We think of $\mathcal{S}$ primarily as a stratification, ie cutting of $\mathbb{C}^n$ into nice pieces. Equivalent to "flats" stratification described above.

→ We think of $\mathcal{C}$ primarily as a combinatorial device (ie a poset), with two extra definitions.

1. If $A, B \in \mathcal{C}$ are faces of the same dimension, we say that A <u>opposes</u> B if they are on opposite sides of a wall of one dim. lower.
2. If $A, B, C \in \mathcal{C}$, we say that they are <u>collinear</u> if we can draw a straight line in $\mathbb{R}^n$ intersecting A, B, C in order.



<u>Perverse sheaves</u>: If X is a top. space, local systems on X $\cong$ finite dim reps. of $\pi_1(X)$; $LS(X)$ are constructible by def.
Perverse sheaves generalise this to stratified spaces.

1. $\mathrm{Perv}(\mathbb{C}^n, \mathcal{S}) \subseteq D^b_c(X, \mathcal{S})$
2. If $\mathcal{U} \in \mathcal{S}$ is the open stratum with $j : \mathcal{U} \hookrightarrow \mathbb{C}^n$, and $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}^n, \mathcal{S})$, then $j^{-1}\mathcal{F} \in LS(\mathcal{U})[n] = \mathrm{Perv}(\mathcal{U}, \mathcal{S}|_{\mathcal{U}})$

③ If $\mathcal{U}$ = open stratum & $V = \mathbb{C}^n \setminus \mathcal{U}$, we have a gluing or recollement as follows:

$$\begin{tikzcd} \mathrm{Perv}(V, \mathcal{S}|_V) \arrow[r, "i_* = i_!"] & \mathrm{Perv}(\mathbb{C}^n, \mathcal{S}) \arrow[l, bend right, "i^{-1}"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^{-1} = j^!"] & LS(\mathcal{U})[n] \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

[Analogue of short exact sequence : $LS(\mathcal{U})[n] \simeq$ quotient category]

<u>Algebra</u> (based on Kapranov-Schechtman):
Since $\mathrm{Perv}(\mathbb{C}^n, \mathcal{S})$ is an abelian category, try to exhibit it as R-mod for some algebra R.

<u>Defn</u>: R is the algebra over $\mathbb{C}$ defined as follows.
It has generators $\{e_C \mid C \in \mathcal{C}\}$

① Each e_C is idempotent : $e_C^2 = e_C$
② If $A \leq B$, then $e_A e_B = e_B = e_B e_A$

$\Rightarrow e_{\{0\}} = 1$

③ If A, B, C are collinear, then $e_A e_B e_C = e_A e_C$
④ If A opposes B, then $e_A e_B e_A + (1 - e_A)$ is invertible.

<u>Thm</u>: $\mathrm{Perv}(\mathbb{C}^n, \mathcal{S}) \simeq$ finite dimε R-mod.

<u>Q</u>: What about the recollement?

<u>Thm</u> (Well-known): If R any ring & $e \in R$ idempotent, then there is a recollement:

$$\begin{tikzcd} R/ReR\text{-mod} \arrow[r] & R\text{-mod} \arrow[l, bend right, "M/ReM \,\mapsfrom\, M"'] \arrow[l, bend left, "\{m \mid eRm = 0\} \,\mapsfrom\, M"] \arrow[r, "M \mapsto eM"] & eRe\text{-mod} \arrow[l, bend right, "Re \otimes_{eRe} - \,=\, \mathrm{Ind}"'] \arrow[l, bend left, "\mathrm{Hom}_{eRe}(eR, -)"] \end{tikzcd}$$

Main results:

Fix any $A \in \mathcal{C}$ with max'l dimension [so its complex span is $\mathcal{U}$]. Let $e := e_A$.

Thm (B.): There is an equivalence of recollements as follows:

$$\begin{tikzcd}
\mathrm{Perv}(V, \mathcal{S}|_V) \arrow[r] & \mathrm{Perv}(\mathbb{C}^n, \mathcal{S}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & LS(\mathcal{U})[n] \arrow[l, bend right] \arrow[l, bend left] \\
R/ReR\text{-mod} \arrow[u] \arrow[r] & R\text{-mod} \arrow[u] \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & eRe\text{-mod} \arrow[u] \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

Thm (B.) If $\mathcal{H}$ comes from a Weyl group, then the equivariant version [based on work of Weissman].

Cor (B.): $\mathbb{C}[\pi_1(\mathcal{U})]$-fdmod $\simeq$ eRe-fdmod. Under certain conditions, we also have $\mathbb{C}[\pi_1(\mathcal{U})] \simeq eRe$ as algebras.