A Thurston compactification of Bridgeland stability space

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A THURSTON COMPACTIFICATION OF BRIDGELAND STABILITY SPACE

Joint with Anand Deopurkar & Anthony Licata

Setup & General Motivation

- $\mathcal{C}$ a triangulated category (e.g. $D^b \operatorname{Rep} A$)

SETUP & GENERAL MOTIVATION

- $\mathcal{C}$ a triangulated category (e.g. $D^b \operatorname{Rep} A$)

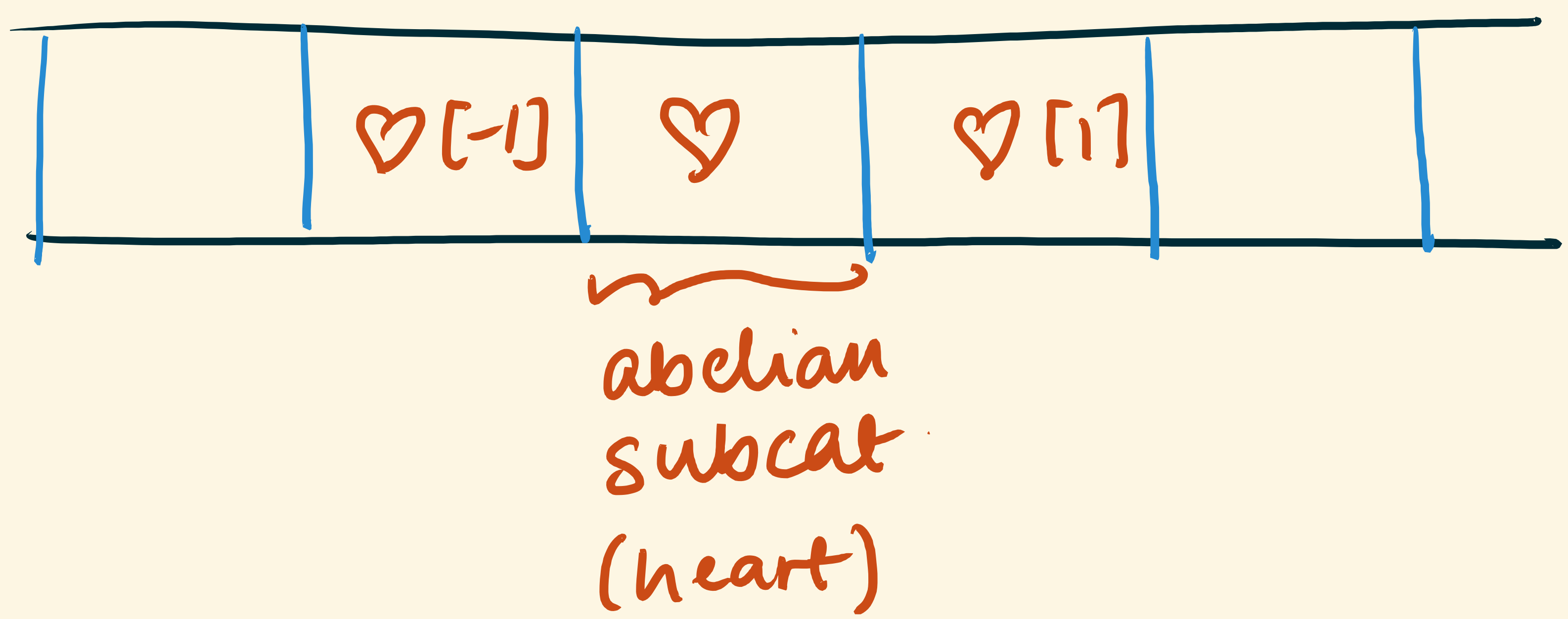
  Equipped with:

  - a shift functor $[1]$
  - distinguished triangles

    $A \to B \to C \to A[1]$

Setup & General Motivation

- $\mathcal{C}$ a triangulated category (e.g. $D^b \mathrm{Rep} A$)
- Often, we have bounded t-structures

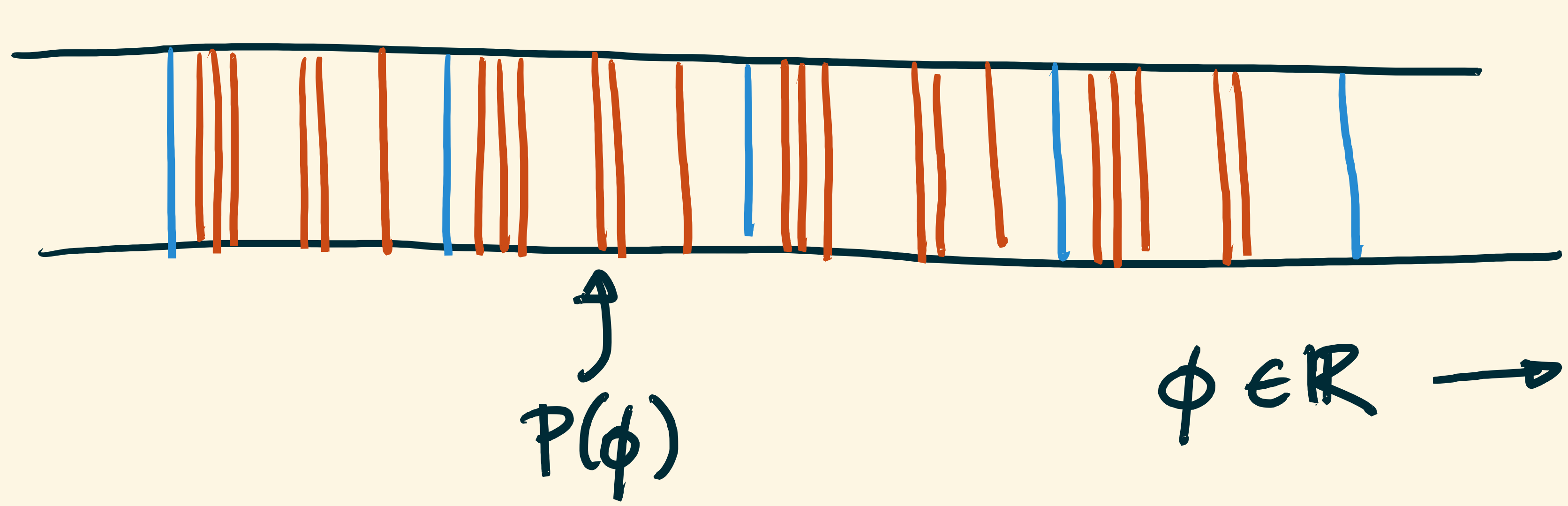


Setup & General Motivation

- $\mathcal{C}$ a triangulated category (e.g. $D^b \operatorname{Rep} A$)
- Often, we have bounded t-structures
- Varying the t-structure is extremely useful, but they don't have a nice parametrization.
- So we turn to Bridgeland stability conditions.

Setup & General Motivation

- A Bridgeland stability condition is roughly an "$\mathbb{R}$-refinement" of a t-structure on $\mathcal{C}$:



Setup & General Motivation

- A Bridgeland stability condition is roughly an "$\mathbb{R}$-refinement" of a t-structure on $\mathscr{C}$.

- Collection of stability conditions on $\mathscr{C}$ forms a nice space (a complex manifold!)

- $\mathrm{Aut}(\mathscr{C}) \circlearrowright \mathrm{Stab}(\mathscr{C}) \leftarrow$ stability space

STABILITY CONDITIONS

- A stability condition on $\mathcal{C}$ consists of :
  - $\heartsuit$, the heart of a bdd t-structure
  - A group homomorphism

    $Z : K(\heartsuit) \to \mathbb{C}$, such that

    $Z(\heartsuit) \subseteq \mathbb{H}$.

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  - A gp homomorphism

    $Z : K(\heartsuit) \to \mathbb{C}$, such that

    $Z(\heartsuit) \subseteq \mathbb{H}$.

- An object $X \in \heartsuit$ is semistable if :

  whenever $Y \subseteq X$, we have

  $$\arg(Z(Y)) \leq \arg(Z(X)).$$

STABILITY CONDITIONS

<u>E.g.</u> $\mathcal{C} = D^b \operatorname{Rep} A_2$. $\qquad \bullet \longrightarrow \bullet$

$S_1 : \quad k \longrightarrow 0 \quad$ (simple)

$S_2 : \quad 0 \longrightarrow k \quad$ (simple)

$E : \quad k \xrightarrow{\sim} k \quad$ (indecomposable)

We have $\quad 0 \to S_2 \to E \to S_1 \to 0$

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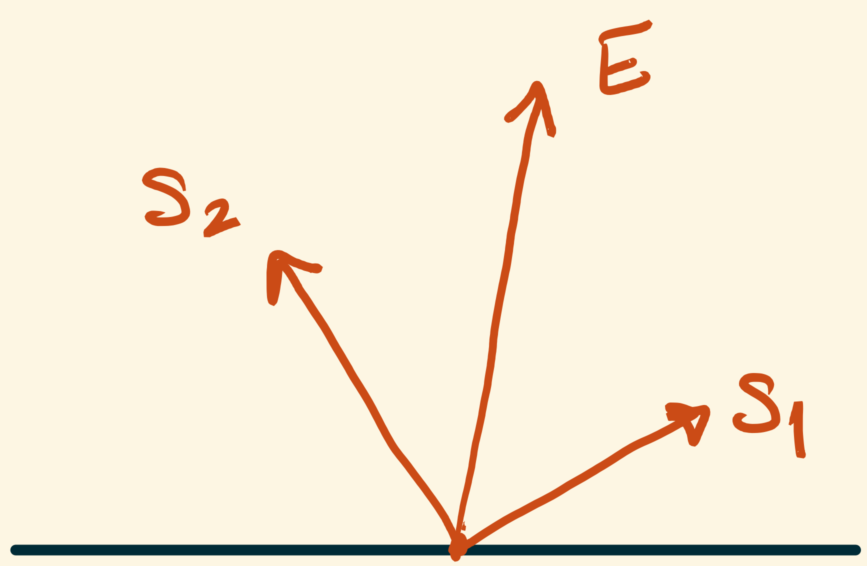
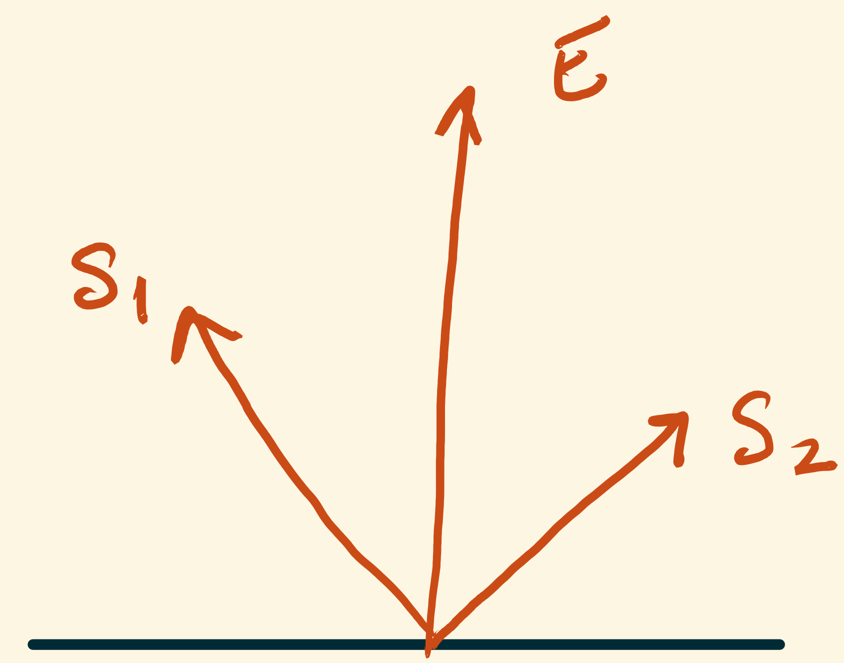
$$0 \to S_2 \to E \to S_1 \to 0$$



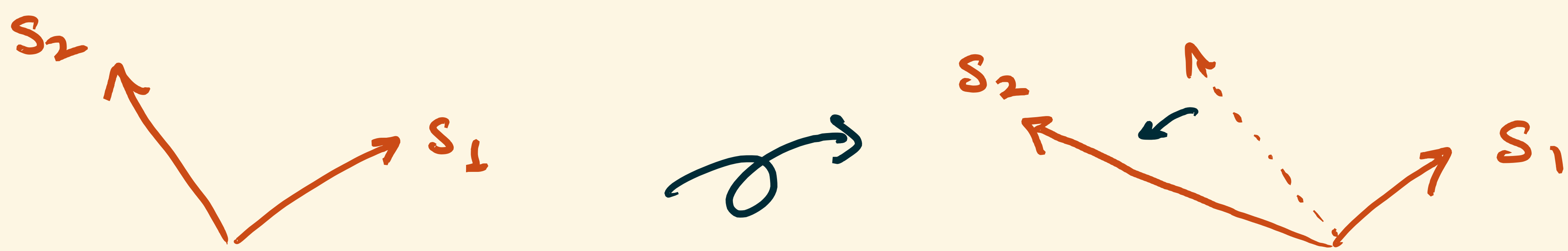
E not semistable



E semistable

PROPERTIES OF STABILITY CONDITIONS

- $(\heartsuit, Z)$ can be deformed continuously by deforming Z. E.g:



PROPERTIES OF STABILITY CONDITIONS

- $(\heartsuit, Z)$ can be deformed continuously by deforming Z.
- The t-structure can (and will!) change in this process.
- $\mathrm{Stab}\,\mathcal{C} := \{(\heartsuit, Z)\}/\mathbb{C}$ is a complex manifold.
- Key property: Harder–Narasimhan filtrations.

HARDER-NARASIMHAN FILTRATIONS

THEOREM (Bridgeland)

If τ is a stability condition, then every $X \in \heartsuit$ has a unique filtration $0 = X_0 \subset X_1 \subset X_2 \cdots \subset X_n = X$, such that

- $A_i = X_i / X_{i-1}$ is semistable, and
- $\arg(A_1) > \arg(A_2) > \cdots > \arg(A_n)$

MAIN EXAMPLE FOR THIS TALK

$\mathcal{C} = K^b \mathrm{Proj}$ of a zigzag algebra

(quotient of path algebra of doubled quiver)

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$\circlearrowleft$

B , the Artin-Tits braid group of quiver

$\Rightarrow$ $B \circlearrowright \mathrm{Stab}(\mathscr{C})$

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B , the Artin-Tits braid group of quiver

$\Rightarrow$ $B \circlearrowright \mathrm{Stab}(\mathcal{C})$

EXAMPLE : Zigzag algebra for type A_2 :

$$\begin{tikzcd} \bullet \arrow[r, bend left, "a"] & \bullet \arrow[l, bend left, "b"] \end{tikzcd} \qquad aba = bab = 0$$

$\mathcal{C}$ generated by P_1 & P_2

MAIN EXAMPLE FOR THIS TALK

$$\mathscr{C} = \langle P_1, P_2 \rangle \qquad\qquad \bullet \rightleftarrows \bullet$$

The objects P_1 & P_2 are <u>spherical</u>:

$$\mathrm{Hom}^k(P_i, P_i) = \begin{cases} \mathbb{C} & \text{for } k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

MAIN EXAMPLE FOR THIS TALK

$\mathscr{C} = \langle P_1, P_2 \rangle$ $\qquad \bullet \rightleftarrows \bullet$

- The objects P_1 & P_2 are spherical.
- Have associated <u>spherical twist</u> autoequivalences

$$\sigma_{P_1} : \mathscr{C} \to \mathscr{C} \quad , \quad \sigma_{P_2} : \mathscr{C} \to \mathscr{C}$$

- These satisfy the braid relation:

$$\sigma_{P_1} \sigma_{P_2} \sigma_{P_1} \simeq \sigma_{P_2} \sigma_{P_1} \sigma_{P_2}$$

$\Rightarrow B_3 \circlearrowright \mathscr{C}$

MAIN QUESTION

Is there a compactification of $\mathrm{Stab}(\mathcal{C})$ such that the action of B extends continuously to the boundary?

STRATEGY FOR COMPACTIFICATION

- Embed $\mathrm{Stab}\,\mathcal{C}$ into an (infinite) projective space, and take closure.

- More precisely:

$$\begin{array}{ccc} \mathrm{Stab}\,\mathcal{C} & \longrightarrow & \mathbb{P}^{S} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \tau & \longmapsto & \left[X \longmapsto \text{"}\tau\text{-mass of } X\text{"}\right]\big/_{\text{scalars}} \end{array}$$

[S = sphericals of $\mathcal{C}$]

STRATEGY FOR COMPACTIFICATION

- $\mathrm{Stab}\,\mathcal{C} \longrightarrow \mathbb{P}^S$ (S = sphericals of $\mathcal{C}$)

  $\tau \mapsto [X \mapsto$ "τ-mass of X"$]/_{\text{scalars}}$

- $m_\tau(X)$ = sum of lengths of $Z(A_i)$, where $A_1, A_2, \dots A_n$ are τ-semistable HN factors.

- Analogous to a construction of Thurston for Teichmüller space.

BOUNDARY POINTS?

Let $\tau \in \mathrm{Stab}\,\mathcal{C}$ & X a spherical object.

Let σ_X = spherical twist in X.

Let m_τ be the image of τ in $\mathbb{P}^S$

PROPOSITION (B-D-L):

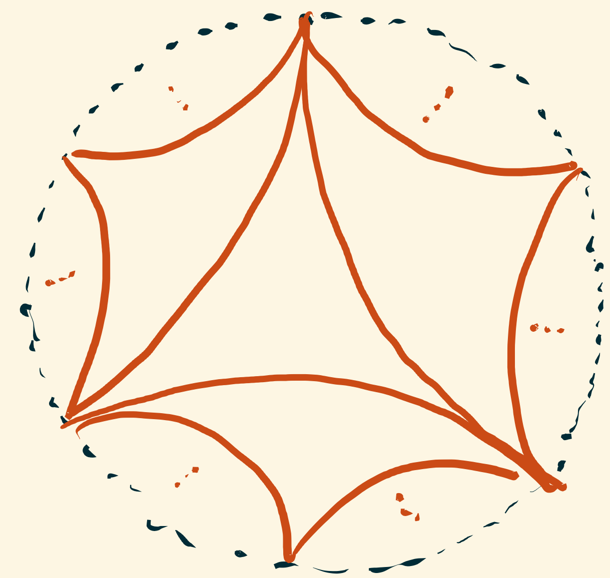
$$\left[\lim_{n\to\infty} m_{\sigma_X^n \tau}\right](Y) = \begin{cases} \dim \mathrm{Hom}(X,Y) & \text{if } X \neq Y \\ 0 & \text{otherwise}, \end{cases}$$

up to a simultaneous scalar.

TYPE A_2 ZIGZAG CASE

THEOREM (Bridgeland–Qiu–Sutherland)

$$\mathrm{Stab}\,\mathcal{C} \simeq$$

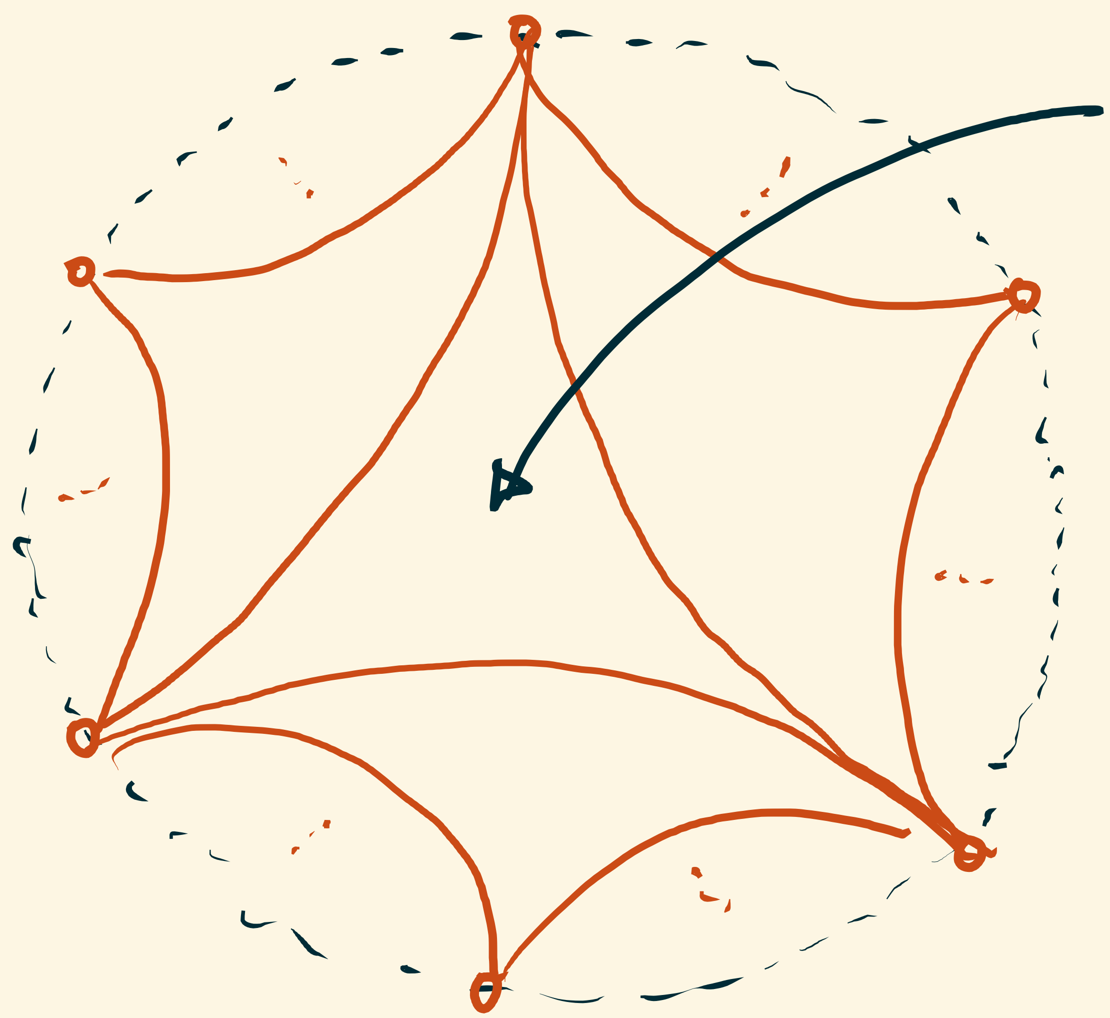


$\circlearrowright B_3$ (via $PSL_2(\mathbb{Z})$)

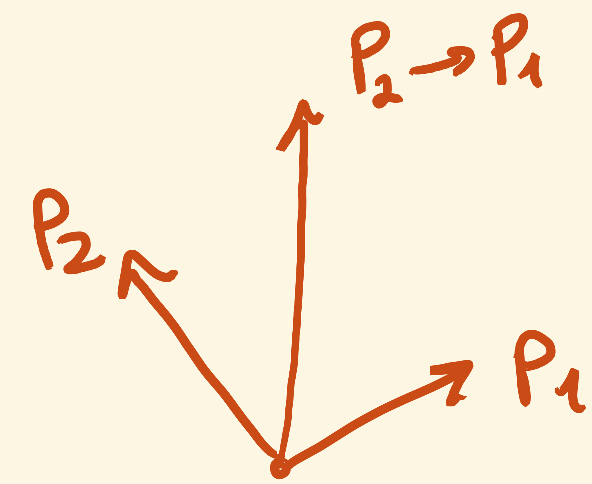
$\circlearrowright$

B_3

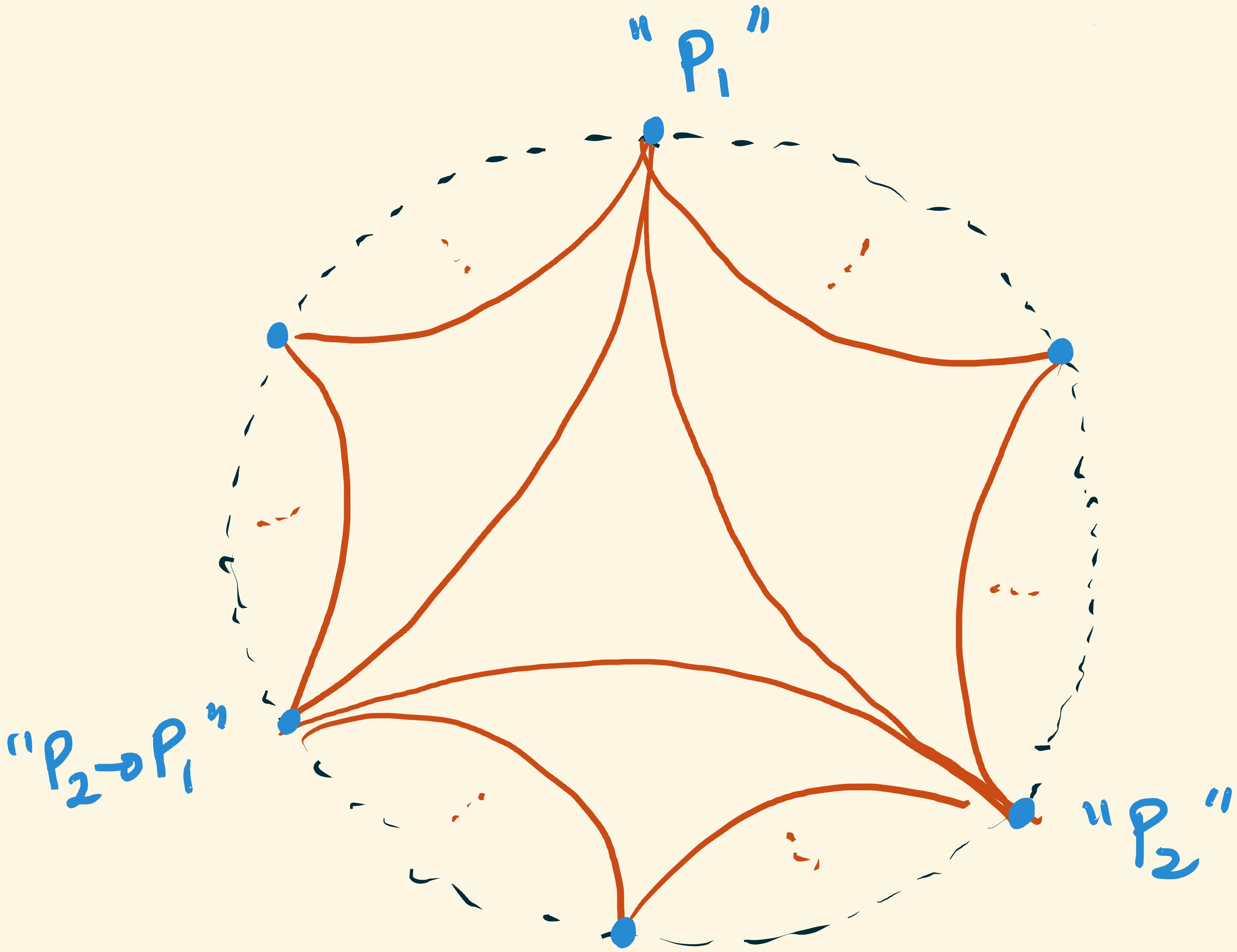
TYPE A_2 ZIGZAG CASE



stability conditions of the form



TYPE A_2 ZIGZAG CASE



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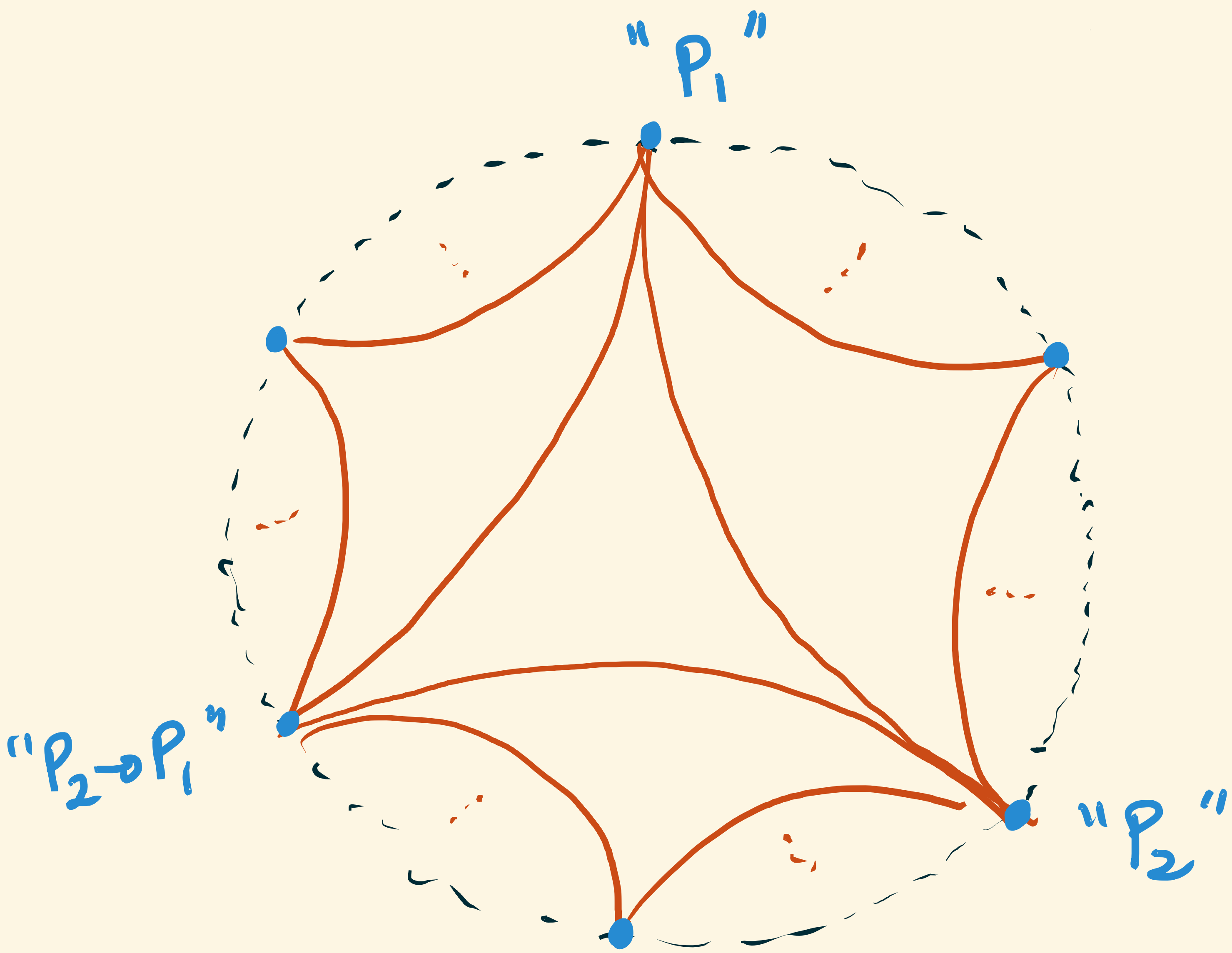
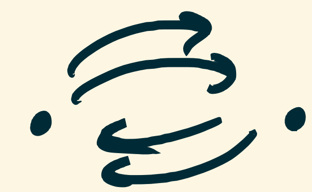


THEOREM (B-D-L)

(1) Our recipe compactifies $\mathrm{Stab}(\mathcal{C})$ to a closed disk

(2) The sphericals (indexed by $\mathbb{Q} \cup \{\infty\}$) are dense in the boundary

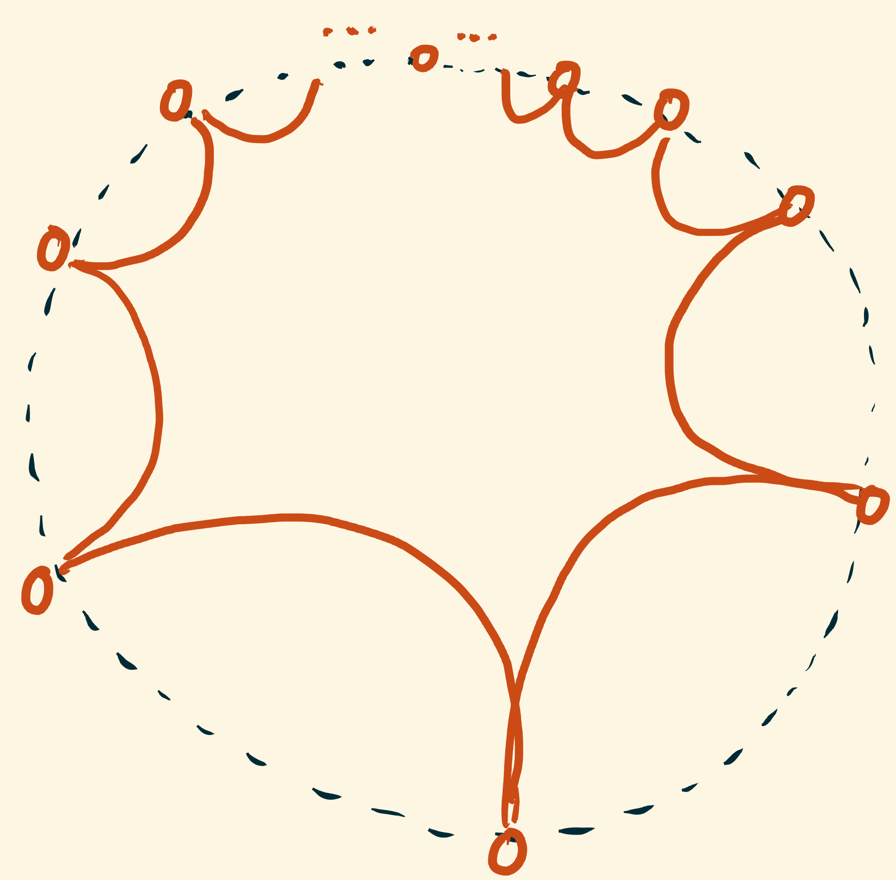
THE $\hat{A}_1$ ZIGZAG CASE



THEOREM (B-D-L)

(1) The closure of Stab $\mathcal{C}$ under the mass map is compact

(2) The spherical objects form a dense subset of the boundary.



PROOF STRATEGY

- Show that $\mathrm{Stab}\,\mathcal{C} \longrightarrow \mathbb{P}^S$ is injective
- Show that its image is compact
- Show homeomorphism onto image
- Compute boundary

PROOF STRATEGY

- Show that $\mathrm{Stab}\,\mathcal{C} \longrightarrow \mathbb{P}^S$ is injective ✓
- Show that its image is compact ✓ any connected quiver (BDL)
- Show homeomorphism onto image
- Compute boundary

PROOF STRATEGY

- Show that $\mathrm{Stab}\,\mathcal{C} \longrightarrow \mathbb{P}^S$ is injective ✓
- Show that its image is compact ✓
- Show homeomorphism onto image → key hard step, achieved via "Harder-Narasimhan automata"
- Compute boundary

HARDER–NARASIMHAN AUTOMATA

- Gadget to encode behaviour of HN filtrations under group actions.

- Simultaneously give:

(1) homeomorphism onto image of

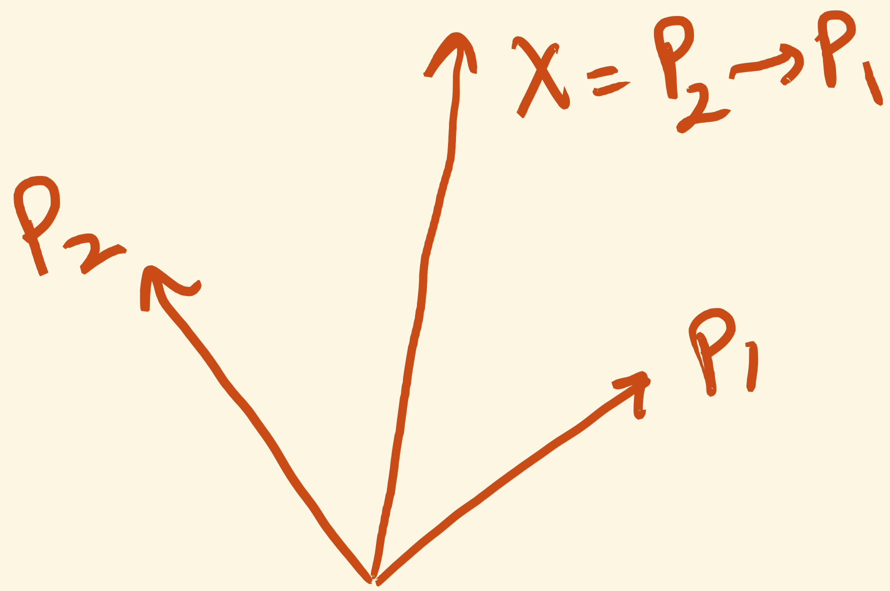
$$\mathrm{Stab}\,\mathcal{C} \to \mathbb{P}^S$$

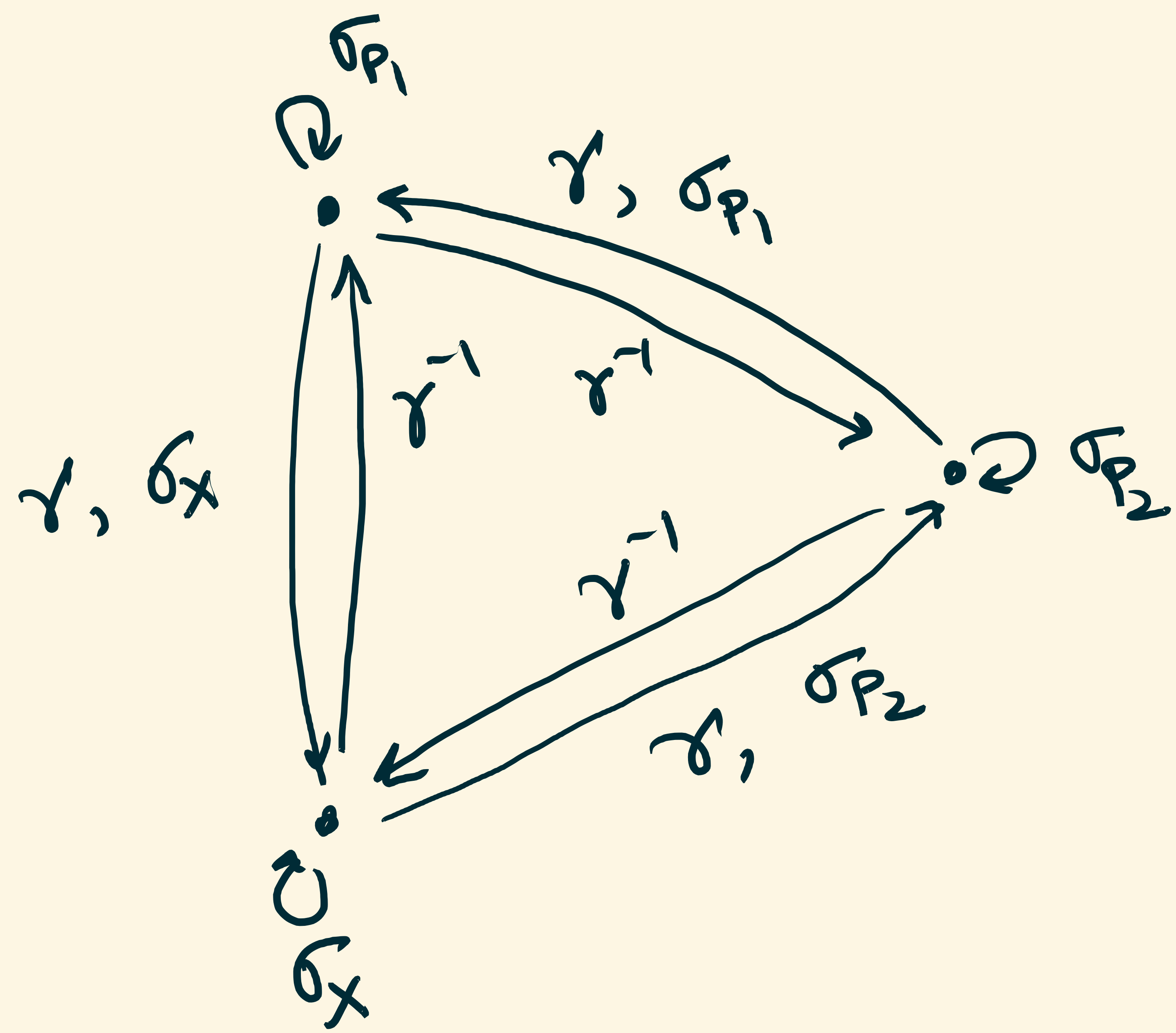
(2) a normal form (solution to word problem) for autoequivalence group

(3) piecewise linear action of group on boundary

HARDER-NARASIMHAN AUTOMATA

A_2 EXAMPLE





FUTURE DIRECTIONS

- Find HN automata for all types
- Compute $\overline{\mathrm{Stab}\,\mathcal{C}}$ in other types
- Deduce results about Artin–Tits groups

THANK YOU!