# A Thurston compactification of stability space


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# The big picture

Group $G \xrightarrow{\text{representation}} G \circlearrowright V$

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$$
\begin{tikzcd}
\text{Group } G \arrow[r] & G \curvearrowright V \arrow[d, "\text{categorification}"] \\
& G \curvearrowright \mathcal{C}
\end{tikzcd}
$$

# The big picture

$$\begin{tikzcd}
\text{Group } G \arrow[rr] & & G \circlearrowright V \arrow[d] \\
& G \circlearrowright \mathrm{Stab} & G \circlearrowright \mathcal{C} \arrow[l, "\text{geometry}"']
\end{tikzcd}$$

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G \circlearrowright \overline{\mathrm{Stab}} & &
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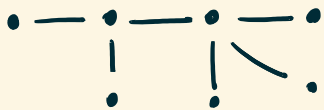
# Setup



$\Gamma$ finite graph — often ADE type

$B_\Gamma$ braid group

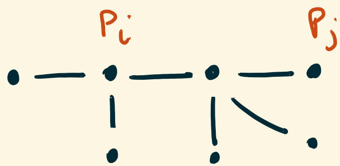
Generators: $\sigma_i$ for $i \in \Gamma$

Relations:

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } i - j$$

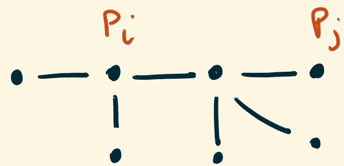
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{otherwise}$$

# Setup




$\mathcal{C}_\Gamma$ triangulated cat. that categorifies geometric rep. of $W_\Gamma$

# Setup




$\mathcal{C}_\Gamma$ triangulated cat. that categorifies geometric rep. of $W_\Gamma$

- $\mathcal{C}_\Gamma$ generated by $P_i$ for $i \in \Gamma$
- Each $P_i$ is spherical:

$$\mathrm{End}^*(P_i) \cong H^*(S^2)$$

# Categorical braid group action

Any $X \in \mathcal{C}_\Gamma$ gives a "twist functor"

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Any $X \in \mathcal{C}_\Gamma$ gives a "twist functor"

$$\sigma_X : \mathcal{C}_\Gamma \longrightarrow \mathcal{C}_\Gamma$$

- $\sigma_X$ equivalence if $X$ spherical
- $\langle \sigma_{P_i} \rangle$ satisfy braid relations,

  giving $B_\Gamma \circlearrowright \mathcal{C}_\Gamma$.

$$
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# Bridgeland stability conditions

Data $(\heartsuit, Z)$ on triangulated cat $\mathcal{C}$

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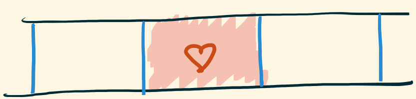
heart of bdd t-structure



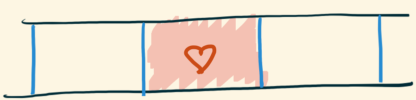
# Bridgeland stability conditions

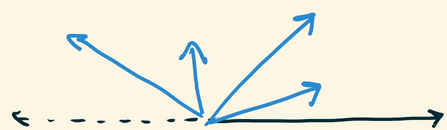
Data $(\heartsuit, Z)$ on triangulated cat $\mathcal{C}$

heart of bdd t-structure

$Z : \heartsuit \to \mathbb{H}$ extending to homomorphism

$Z : K(\mathcal{C}) \to \mathbb{C}$.





# Bridgeland stability conditions

$$\mathcal{C} \longmapsto \mathrm{Stab}(\mathcal{C}) = \{(\heartsuit, Z)\}/\sim$$

Triangulated category $\rightsquigarrow$ Complex manifold [Bridgeland]

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Triangulated category $\rightsquigarrow$ Complex manifold [Bridgeland]

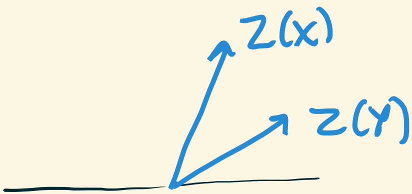
We have :

$$\boxed{B_\Gamma \circlearrowright \mathscr{C}_\Gamma \longmapsto \operatorname{Stab}\mathscr{C}_\Gamma \circlearrowleft B_\Gamma}$$

# Bridgeland stability conditions

Let $\tau = (\heartsuit, Z) \in \mathrm{Stab}\, \mathcal{C}$

$X \in \heartsuit$ is $\tau$-stable if




for all subs $Y \subsetneq X$

[Shifts of stables are also called stable]

# Bridgeland stability conditions

Let $\tau \in \mathrm{Stab}\,\mathcal{C}$. Any $X \in \mathcal{C}$ has a canonical finite filtration with $\tau$-stable subquotients.

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The $\tau$-mass of $X$ is:

$$m_\tau(X) = \sum |Z(A_i)|$$

stable subquotients

$$
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# Towards $\overline{\mathrm{Stab}\, \mathcal{C}_\Gamma}$

Theorem (B-Deopurkar-Licata)

Let $\Gamma$ be connected. Any $\tau$ can be uniquely reconstructed from the mass vector

$$\langle m_\tau(X) \mid X \text{ spherical in } \mathcal{C}_\Gamma \rangle / \text{scaling}$$

# Towards $\overline{\mathrm{Stab}\,\mathcal{C}_\Gamma}$

Let $S$ = sphericals of $\mathcal{C}_\Gamma$. Then:

$$\mathrm{Stab}\,\mathcal{C}_\Gamma \hookrightarrow \mathbb{P}\mathbb{R}^{S}$$

$$\tau \longmapsto \langle m_\tau(X) \rangle$$

# Towards $\overline{\mathrm{Stab}\,\mathcal{C}_\Gamma}$

## Conjectures:

- Closure of Stab in $\mathbb{P}\mathbb{R}^S$ is a closed manifold with boundary (closed ball?)
- $\exists$ faithful piecewise linear action of $B_\Gamma$ on boundary.

## What we know

$\mathrm{Stab}\,\mathcal{C}_\Gamma \hookrightarrow \mathbb{P}\mathbb{R}^S$

**Theorem** [B-D-L]

- Closure is compact for any $\Gamma$
- All conjectures hold for $A_2$ & $\hat{A}_1$

# Boundary of Stab $\mathcal{C}_\Gamma$

Also have $\boxed{S \hookrightarrow \mathbb{P}\mathbb{R}^S}$

$$A \longmapsto \langle \hom(A, X) \rangle /_\sim$$

## Boundary of $\mathrm{Stab}\,\mathcal{C}_\Gamma$

$$S \hookrightarrow \mathbb{P}\mathbb{R}^S \hookleftarrow \mathrm{Stab}\,\mathcal{C}_\Gamma$$

Theorem [B-D-L]

- $S \subset \overline{\mathrm{Stab}\,\mathcal{C}_\Gamma}$
- If $\Gamma$ is ADE, $B_\Gamma \circlearrowright S$ transitively.

(Expect $S$ to be dense in boundary.)

$$
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G \curvearrowright \mathrm{Stab} \arrow[d] & & G \curvearrowright \mathcal{C} \arrow[ll] \\
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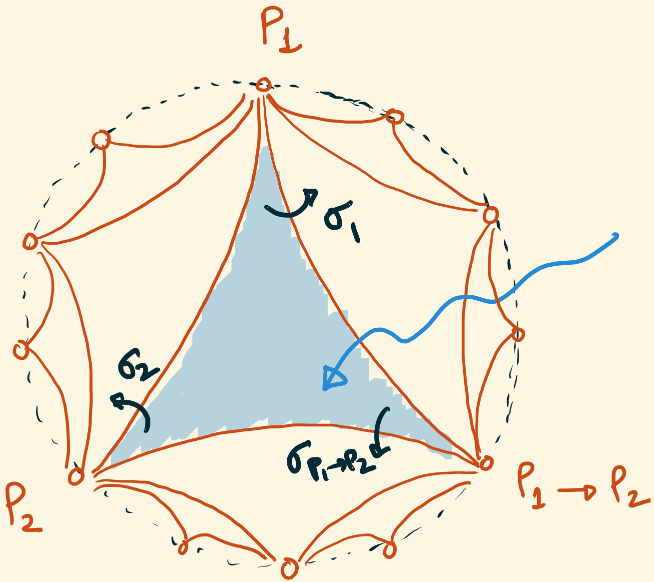
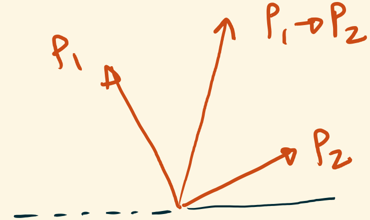
# Illustration : Type $A_2$




♡ = standard heart

$Z$ :



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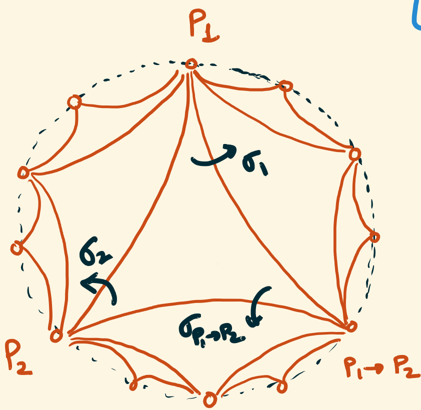



[Thomas, Bridgeland–Qiu–Sutherland]

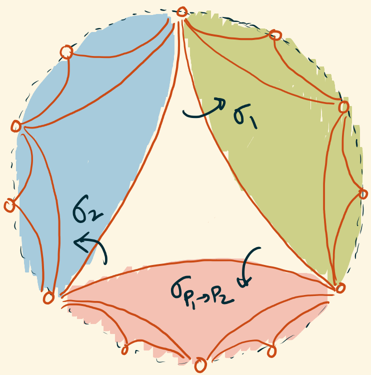
Stab $\simeq$ open disk

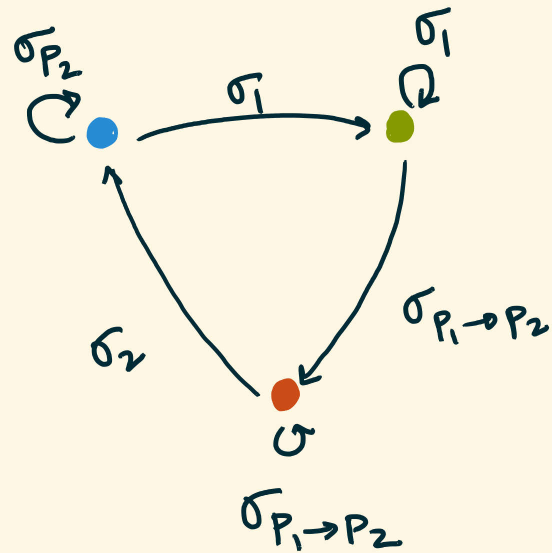
Theorem [B-D-L]

- $\overline{\text{Stab}} \simeq$ closed disk
- Sphericals dense in boundary.

# Illustration : Type $A_2$

PL action of $B_r$ : every arrow is linear



Thank you!