A Thurston compactification for stability space

References:

1) B-Deopurkar-Licata
2) Louis Becker honours thesis

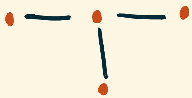
Plan

$$\left.\begin{tikzcd} & \text{CY2 category } \mathcal{C} \arrow[d] & \\ & \operatorname{Stab}\mathcal{C} \arrow[dl] \arrow[dr] & \\ \overline{\operatorname{Stab}}\,\mathcal{C} \arrow[rr] \arrow[d] \arrow[dr] & & \overline{\operatorname{Stab}}_q\,\mathcal{C} \\ \text{“Sphere of sphericals”} & \text{HN automata.} & \end{tikzcd}\right\} \circlearrowright B_{\mathcal{C}}$$

Plan

$$
\begin{tikzcd}
 & \text{CY2 category } \mathcal{C} \arrow[d] & \\
 & \operatorname{Stab}\mathcal{C} \arrow[dl] \arrow[dr] & \\
\overline{\operatorname{Stab}}\,\mathcal{C} \arrow[rr] \arrow[d] \arrow[drr] & & \overline{\operatorname{Stab}}_q\,\mathcal{C} \\
\text{"Sphere of sphericals"} & & \text{HN automata.}
\end{tikzcd}
$$

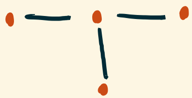
Our category



Γ a connected graph

$\mathcal{C}_\Gamma$ triangulated category generated by $\{P_i \mid i \in \Gamma\}$.

Our category



Γ a connected graph

$\mathcal{C}_\Gamma$ triangulated category generated by $\{P_i \mid i \in \Gamma\}$.

$n \rightarrow$	0	1	2
$\hom^n(P_i, P_i)$	1	0	1
$\hom^n(P_i, P_j)$	0	1	0

if 

Facts

1) $\mathcal{C}_\Gamma$ is CY 2 : $\mathrm{Hom}^\bullet(X,Y) \simeq \mathrm{Hom}^\bullet(Y, X[2])^\vee$

2) Each P_i is spherical.

Facts

1) $\mathcal{C}_\Gamma$ is CY2 : $\mathrm{Hom}^\bullet(X,Y) \simeq \mathrm{Hom}^\bullet(Y, X[2])^\vee$

2) Each P_i is spherical.

3) The extension closure of $\{P_i\}$ is the heart of a bounded t-structure on $\mathcal{C}_\Gamma$

Facts

1) $\mathcal{C}_\Gamma$ is CY 2 : $\mathrm{Hom}^\bullet(X,Y) \simeq \mathrm{Hom}^\bullet(Y, X[2])^\vee$

2) Each P_i is spherical.

3) The extension closure of $\{P_i\}$ is the heart of a bounded t-structure on $\mathcal{C}_\Gamma$

"standard heart"

Facts

1) $\mathcal{C}_\Gamma$ is CY 2 : $\mathrm{Hom}^\bullet(X,Y) \simeq \mathrm{Hom}^\bullet(Y, X[2])^\vee$

2) Each P_i is spherical.

3) The extension closure of $\{P_i\}$ is the heart of a bounded t-structure on $\mathcal{C}_\Gamma$

"standard heart"

* We'll focus on $\Gamma = A_n$ 

Example

$\Gamma = A_2 \qquad \bullet\!-\!\!\!-\!\!\!-\!\bullet$

$\mathcal{C}_\Gamma$ has P_1 & P_2 with

$$\hom^1(P_1, P_2) = \hom^1(P_2, P_1) = 1$$

$\Rightarrow \exists!$ non-split exact triangle

$$P_2 \to * \to P_1 \to P_2[1]$$

Example

$\Gamma = A_2$ 

$\mathcal{C}_\Gamma$ has P_1 & P_2 with

$$\hom^1(P_1, P_2) = \hom^1(P_2, P_1) = 1$$

$\Rightarrow \exists!$ non-split exact triangle

$$P_2 \to * \to P_1 \to P_2[1]$$

call this "$P_1 \to P_2$"

Similarly, we have "$P_2 \to P_1$"

Spherical twists

Let $X \in \mathcal{C}_\Gamma$ be a spherical object.

We have a functor, the <u>spherical twist in X</u> [Seidel-Thomas]:

$$\sigma_X : \mathcal{C}_\Gamma \to \mathcal{C}_\Gamma \quad \text{with}$$

$$\sigma_X = \mathrm{Cone}\left(\mathrm{Hom}(X,Y) \otimes X \xrightarrow{ev} Y\right)$$

Facts

1) σ_X is an autoequivalence

Facts

1) σ_X is an autoequivalence

2) $\{\sigma_{P_i}\}$ satisfy braid relations

$\Rightarrow$ braid group of Γ acts on $\mathcal{C}_\Gamma$

Facts

1) σ_X is an autoequivalence

2) $\{\sigma_{P_i}\}$ satisfy braid relations

3) If Y spherical, then $\sigma_X(Y)$ spherical

Facts

1) σ_X is an autoequivalence

2) $\{\sigma_{P_i}\}$ satisfy braid relations

3) If Y spherical, then $\sigma_X(Y)$ spherical

4) If Γ is type ADE then all sphericals are in one orbit under the spherical twist group.

<u>Example</u> : $\bullet$ —— $\bullet$ $\quad \Gamma = A_2$

$$\sigma_{P_1}(P_1) = P_1[-1] \qquad \sigma_{P_2}(P_1) = P_2 \to P_1$$

$$\sigma_{P_1}(P_2) = P_1 \to P_2 \qquad \sigma_{P_2}(P_2) = P_2[-1]$$

$\Rightarrow$ $P_1 \to P_2$, $P_2 \to P_1$ are spherical.

Example : $\bullet$ — $\bullet$ $\quad \Gamma = A_2$

$$\sigma_{P_1}(P_1) = P_1[-1] \qquad \sigma_{P_2}(P_1) = P_2 \to P_1$$

$$\sigma_{P_1}(P_2) = P_1 \to P_2 \qquad \sigma_{P_2}(P_2) = P_2[-1]$$

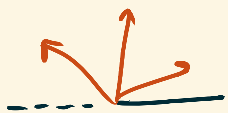
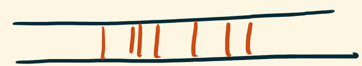
Theme in the remainder of the talk: understand $\operatorname{Stab} \mathcal{C}_\Gamma$ via the spherical objects & twist group action on them

Plan

$$\begin{tikzcd}
 & \text{CY2 category } \mathcal{C} \arrow[d] & \\
 & \mathrm{Stab}\,\mathcal{C} \arrow[dl] \arrow[dr] & \\
 \overline{\mathrm{Stab}}\,\mathcal{C} \arrow[rr] \arrow[d] \arrow[dr] & & \overline{\mathrm{Stab}}_q\,\mathcal{C} \\
 \text{``Sphere of sphericals''} & \text{HN automata.} &
\end{tikzcd}$$

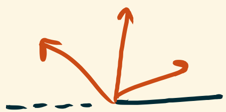
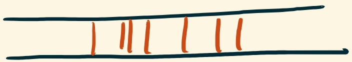
Stability conditions & stability space

A stability condition on $\mathcal{C}_\Gamma$ consists of a compatible pair

Z :  and P : 

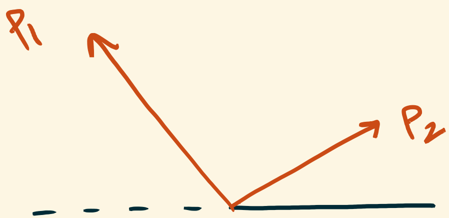
Stability conditions & stability space

A stability condition on $\mathcal{C}_\Gamma$ consists of a compatible pair

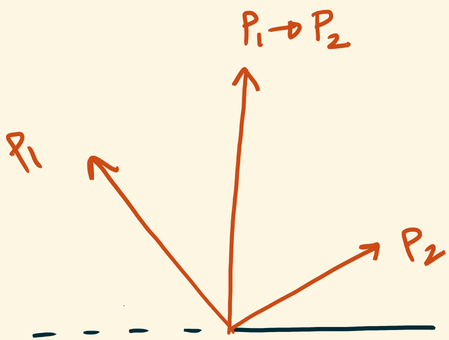
Z :  and P : 

Set $\mathrm{Stab}\,\mathcal{C}_\Gamma$ to be the space of (Z, P) modulo natural $\mathbb{C}$-action

Example : $\Gamma = A_2$ 

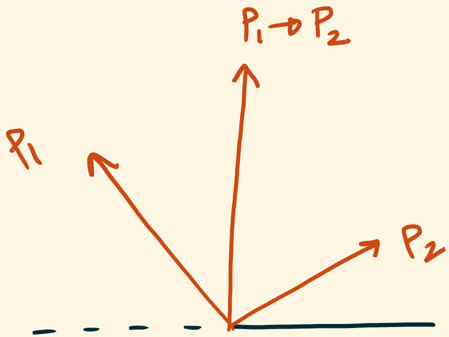


Example: $\Gamma = A_2$ $\quad \bullet \!-\!\!-\!\!-\! \bullet$



These are the stable objects (up to shift)

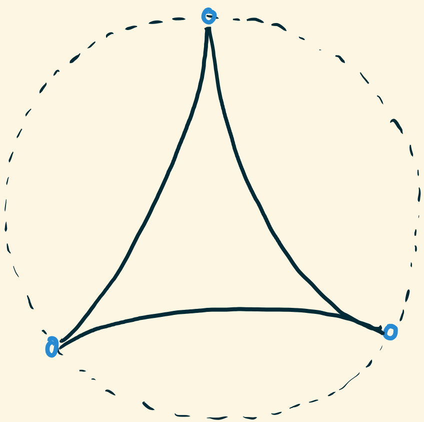
Example: $\Gamma = A_2$



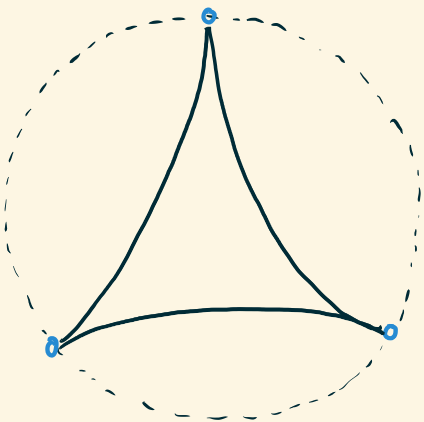
These are the stable objects (up to shift)

$*$ $\tau \in \mathrm{Stab}$ determined uniquely by three sides of a Δ.

Stab $\mathscr{C}_{A_2}$



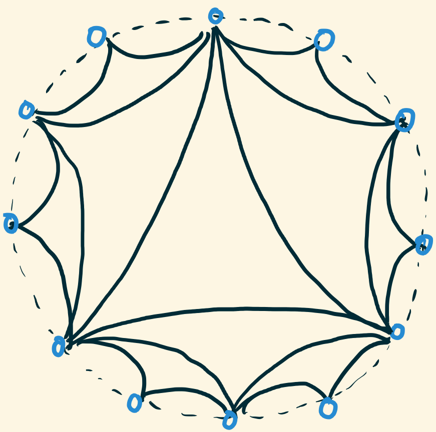
Stab $\mathcal{C}_{A_2}$



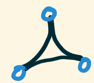
Theorem [Thomas, Bridgeland–Qiu–Sutherland]

$\operatorname{Stab} \mathcal{C}_{A_2} \simeq$ open disk

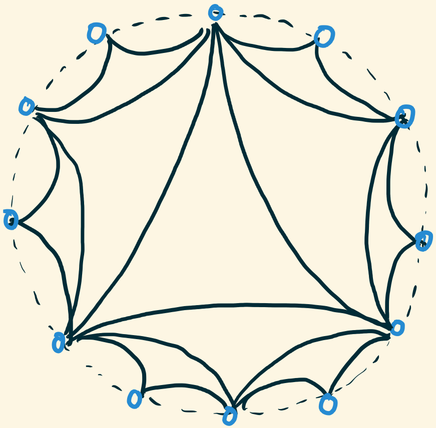
Stab $\mathcal{C}_{A_2}$



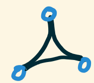
Theorem [Thomas, Bridgeland-Qiu-Sutherland]

Stab $\mathcal{C}_{A_2} \simeq$ open disk, tiled by images of  under braid group action (via $PSL_2(\mathbb{Z})$).

Stab $\mathcal{C}_{A_2}$



<u>Theorem</u> [Thomas, Bridgeland-Qiu-Sutherland]

Stab $\mathcal{C}_{A_2} \cong$ open disk, tiled by images of  under braid group action (via $PSL_2(\mathbb{Z})$).

<u>Q</u> : How to fill in the boundary?

How to compactify?

- Intrinsic approach: degenerations of stability conditions (see Bolognese for partial answer)

How to compactify?

- Intrinsic approach: degenerations of stability conditions (see Bolognese for partial answer)

- Extrinsic approach (following Thurston): Embed

$$\mathrm{Stab}\,\mathcal{C}_r \hookrightarrow \mathbb{P}^\infty,$$

then take closure & interpret boundary.

Plan

$$\begin{tikzcd}
 & \text{CY2 category } \mathcal{C} \arrow[d] & \\
 & \mathrm{Stab}\,\mathcal{C} \arrow[dl] \arrow[dr] & \\
 \overline{\mathrm{Stab}\,\mathcal{C}} \arrow[rr] \arrow[d] \arrow[dr] & & \overline{\mathrm{Stab}_q\,\mathcal{C}} \\
 \text{"Sphere of sphericals"} & \text{HN automata.} &
\end{tikzcd}$$

HN filtrations & HN mass

Let τ be a stability condition.

$$m_\tau(X) := \begin{cases} |Z(X)| & \text{if } X \ \tau\text{-semistable} \\ \sum m_\tau(X_i) & \text{if } \{X_i\} \text{ the HN factors of } X. \end{cases}$$

HN filtrations & HN mass

Let τ be a stability condition.

Fix $q>0$.

$$m_{\tau,q}(X) := \begin{cases} q^{\phi(X)}|Z(X)| & \text{if } X \ \tau\text{-semistable} \\ \sum m_{\tau,q}(X_i) & \text{if } \{X_i\} \text{ the HN factors of } X. \end{cases}$$

HN filtrations & HN mass

Let τ be a stability condition.

Fix $q>0$.

$$m_{\tau,q}(X) := \begin{cases} q^{\phi(X)}|Z(X)| & \text{if } X \ \tau\text{-semistable} \\ \sum m_{\tau,q}(X_i) & \text{if } \{X_i\} \text{ the HN factors of } X. \end{cases}$$

* $m_{\tau,q}(X)$ "measures" X with respect to τ.

$q=1$ $\qquad$ $q>0$

<u>Theorem</u> [BDL, Becker]

Let Γ be connected.

Then $\tau \in \mathrm{Stab}\,\mathcal{C}$ can be recovered from

$$\langle m_{\tau,q}(X) \mid X \text{ spherical} \rangle$$

$q=1$ $q>0$

<u>Theorem</u> [BDL, Becker]

Let Γ be connected.

Then $\tau \in \operatorname{Stab}\mathcal{C}$ can be recovered from

$$\langle m_{\tau,q}(X) \mid X \text{ spherical} \rangle$$

Let S = sphericals of $\mathcal{C}_\Gamma$. Define

$$m_{\tau,q} : \operatorname{Stab}\mathcal{C}_\Gamma \longrightarrow \mathbb{PR}^S$$

$$\tau \longmapsto \langle m_{\tau,q}(X) \mid X \in S \rangle / \sim$$

Let S = sphericals of $\mathscr{C}_\Gamma$. Define

$$m_{\tau,q} : \operatorname{Stab} \mathscr{C}_\Gamma \longrightarrow \mathbb{P}\mathbb{R}^S$$
$$\tau \longmapsto \langle m_{\tau,q}(x) \mid x \in S \rangle / \sim$$

Theorem $\Rightarrow$ $m_{\tau,q}$ is injective.

Set $\overline{\operatorname{Stab} \mathscr{C}_\Gamma}$ = closure of image.

Boundary points?

Theorem [BDL, Becker]. Let $X \in \mathcal{S}$

$$\left(\lim_{n \to \infty} m_{\sigma_X^n \tau, q} \right) = \langle \hom(X, Y) \mid Y \in \mathcal{S} \rangle / \sim$$

Boundary points?

Theorem [BDL, Becker]. Let $X \in \mathcal{S}$

$$\left(\lim_{n\to\infty} m_{\sigma_X^n \tau, q}\right) = \langle \hom(X, Y) \mid Y \in \mathcal{S} \rangle /_\sim$$

$$\Rightarrow \boxed{\mathcal{S} \subseteq \overline{\mathrm{Stab}\, \mathcal{C}_\Gamma}}$$

$$X \longmapsto \langle \hom(X, Y) \mid Y \in \mathcal{S} \rangle /_\sim$$

Some conjectures $[g=1]$

- $\overline{\mathrm{Stab}\,\mathcal{C}_\Gamma}$ = closed manifold w/ boundary
- $\overline{\mathrm{Stab}\,\mathcal{C}_\Gamma} \simeq$ closed ball
- S is dense in the boundary sphere

Some conjectures [$q=1$]

- $\overline{\mathrm{Stab}\,\mathcal{C}_\Gamma}$ = closed manifold w/ boundary
- $\overline{\mathrm{Stab}\,\mathcal{C}_\Gamma} \cong$ closed ball
- S is dense in the boundary sphere

$\to$ proved for $\Gamma = A_2, \hat{A}_1$ [BDL]

Some conjectures [$q \neq 1$]

- $\overline{\mathrm{Stab}\,\mathcal{C}_\Gamma}$ = closed manifold w/ boundary
- $\overline{\mathrm{Stab}\,\mathcal{C}_\Gamma} \simeq$ closed ball
- ~~S is dense in the boundary sphere~~

Some conjectures [9+1]

- $\overline{\mathrm{Stab}\,\mathcal{C}_\Gamma}$ = closed manifold w/ boundary
- $\overline{\mathrm{Stab}\,\mathcal{C}_\Gamma} \simeq$ closed ball
- ~~S is dense in the boundary sphere~~

→ proved for $\Gamma = A_2$ [Becker]

+ a thickened version of S is dense in the boundary sphere.

Plan

$$
\begin{tikzcd}
 & & \text{CY2 category } \mathcal{C} \arrow[d] & \\
 & & \mathrm{Stab}\,\mathcal{C} \arrow[dl] \arrow[dr] & \\
 & \overline{\mathrm{Stab}}\,\mathcal{C} \arrow[rr] \arrow[dl] \arrow[dr] & & \overline{\mathrm{Stab}}_q\,\mathcal{C} \\
\text{"Sphere of sphericals"} & & \text{HN automata.} &
\end{tikzcd}
$$

HN dynamics of boundary sphericals

* Consider $\Gamma = A_n$ & $q = 1$. Fix some τ.

HN supports of sphericals + behaviour under spherical twists is well-behaved.

HN dynamics of boundary sphericals

* Consider $\Gamma = A_n$ & $q = 1$. Fix some τ.

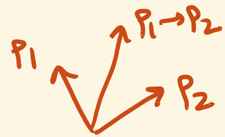
HN supports of sphericals + behaviour under spherical twists is well-behaved.

$\{\Sigma \mid \Sigma = \text{HN support of some spherical}\}$

is constrained.

$\{\Sigma \mid \Sigma = \text{HN support of some spherical}\}$ is constrained.

<u>Example</u> :



$$\Sigma = \{P_2, P_1\}, \quad \{P_1, P_1 \to P_2\}, \quad \{P_1 \to P_2, P_2\}$$

& subsets.

Facts [in progress!]

1) A spherical object can be uniquely reconstructed from its HN support & multiplicities

Facts [in progress!]

1) A spherical object can be uniquely reconstructed from its HN support & multiplicities

2) Construct a simplicial complex w/ simplices the possible Σ. Its geometric realisation is a sphere in which the sphericals are dense.

Facts [in progress!]

1) A spherical object can be uniquely reconstructed from its HN support & multiplicities

2) Construct a simplicial complex w/ simplices the possible Σ. Its geometric realisation is a sphere in which the sphericals are dense.

expect to realise this as $\partial \overline{\mathrm{Stab}\,\mathcal{C}_r}$

Facts [in progress!]

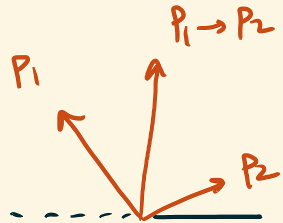
1) A spherical object can be uniquely reconstructed from its HN support & multiplicities

2) Construct a simplicial complex w/ simplices the possible Σ. Its geometric realisation is a sphere in which the sphericals are dense.

3) A wall-cross $\tau \leftrightarrow \tau'$ induces piecewise linear homeomorphisms between these spheres.

Example :

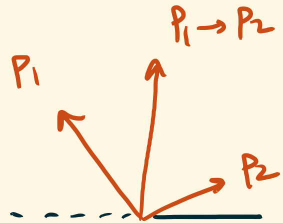
$T = A_2$, τ :



Maximal simplices: $\{P_2, P_1\}$, $\{P_1, P_1 \to P_2\}$, $\{P_1 \to P_2, P_2\}$

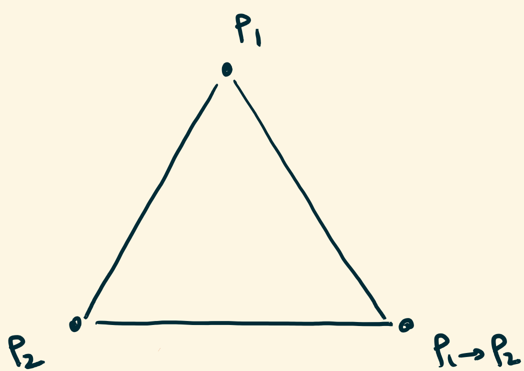
Example :

$T = A_2$, τ :



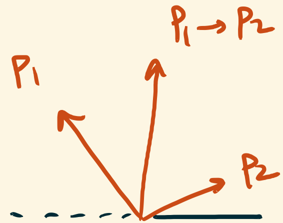
Maximal simplices: $\{P_2, P_1\}$, $\{P_1, P_1 \to P_2\}$, $\{P_1 \to P_2, P_2\}$

Geometric realisation:



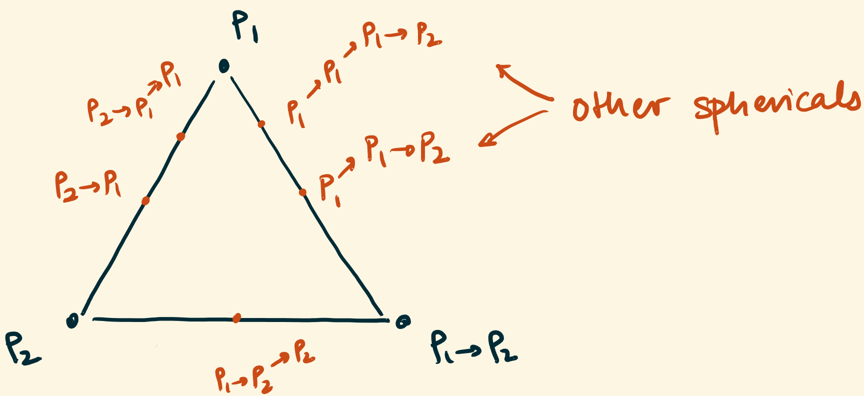
Example :

$T = A_2$, τ :



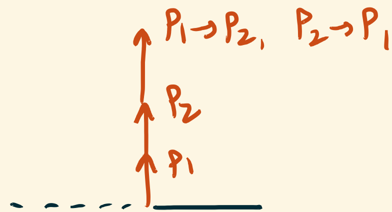
Maximal simplices: $\{P_2, P_1\}$, $\{P_1, P_1 \to P_2\}$, $\{P_1 \to P_2, P_2\}$

Geometric realisation:



Example :

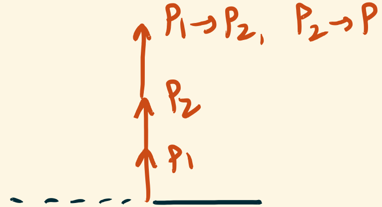
$T = A_2$, τ :



Max simplices: $\{P_1, P_1 \to P_2\}$, $\{P_1, P_2 \to P_1\}$, $\{P_2, P_1 \to P_2\}$, $\{P_2, P_2 \to P_1\}$

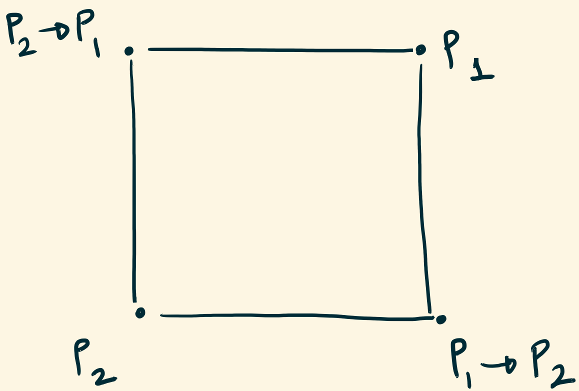
Example :

$T = A_2$, τ :



Max simplices: $\{P_1, P_1 \to P_2\}$, $\{P_1, P_2 \to P_1\}$, $\{P_2, P_1 \to P_2\}$, $\{P_2, P_2 \to P_1\}$

Geometric realisation:

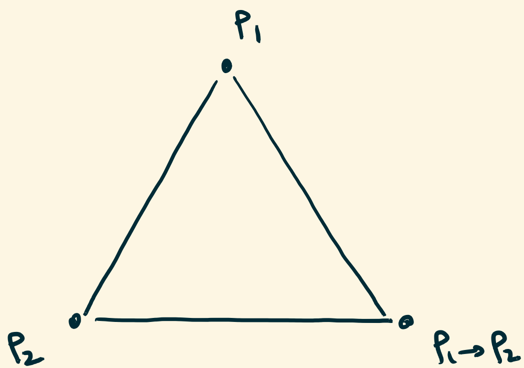


Plan

$$
\begin{tikzcd}
 & \text{CY2 category } \mathcal{C} \arrow[d] & \\
 & \mathrm{Stab}\,\mathcal{C} \arrow[dl] \arrow[dr] & \\
\overline{\mathrm{Stab}}\,\mathcal{C} \arrow[rr] \arrow[d] \arrow[dr] & & \overline{\mathrm{Stab}}_q\,\mathcal{C} \\
\text{“Sphere of sphericals”} & \text{HN automata.} &
\end{tikzcd}
$$

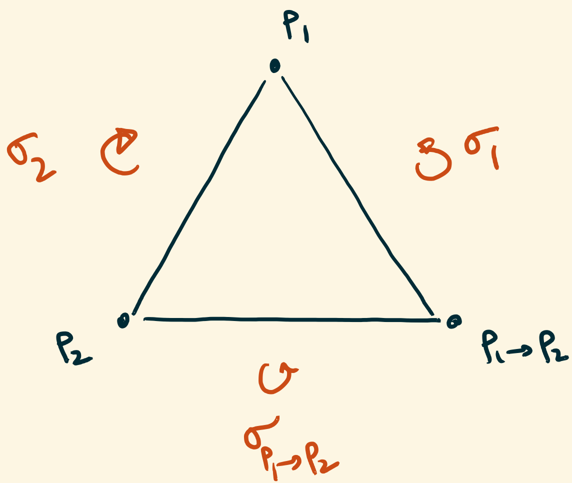
HN automaton (type A_2)

Sometimes, we have more!

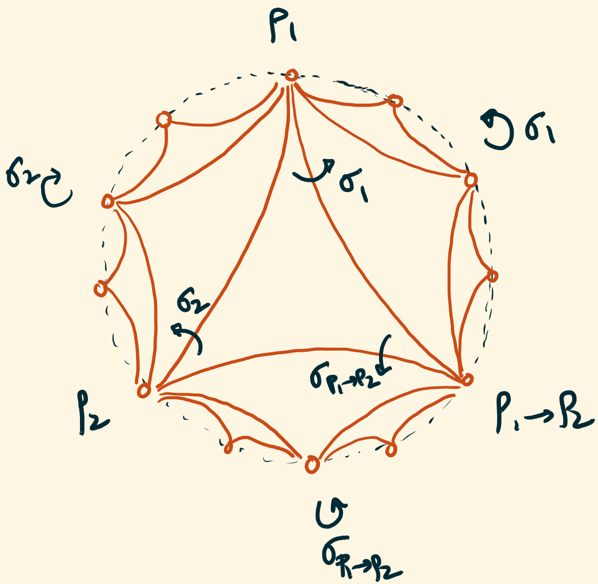


HN automaton (type A_2)

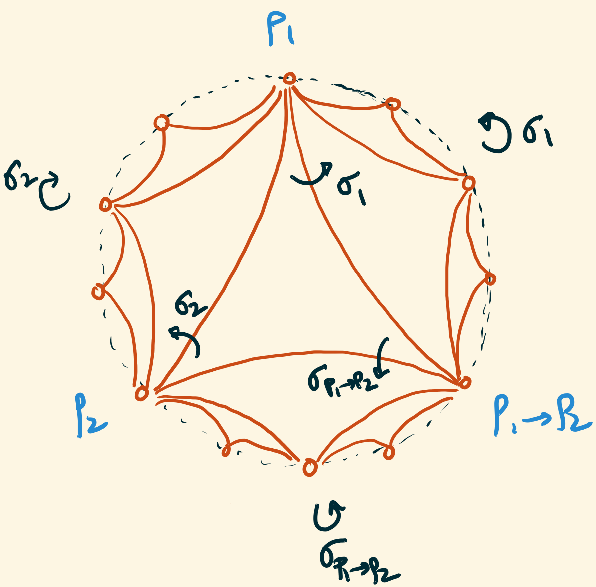
Sometimes, we have more! Spherical twists give linear maps between simplices that control dynamics of HN factors.

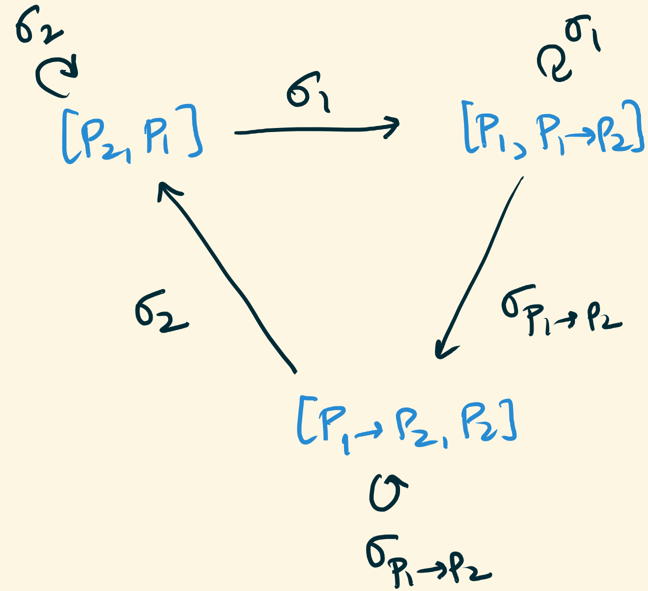


HN automaton (type A_2)



HN automaton (type A_2)





What next?

- A "sphere of sphericals" for A_n & $\hat{A}_n$ (in progress)
- HN automata for $A_n, \hat{A}_n$, more generally
- Understand the remaining points of $\overline{\mathrm{Stab}\, \mathcal{C}_\Gamma}$ categorically
- What happens for $q \neq 1$?

Thank you!