Bridgeland stability conditions & autoequivalences.

* **Plan**

① Introduce category & autoequivalences

[Q: How do objects evolve? How quickly do they grow? Is the action transitive?]

② Bridgeland stability conditions – What do they give us?

③ Applications; evolution [automata]
simplification ["stripping off"]

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① $\mathcal{C}$ and $\mathrm{Aut}(\mathcal{C})$

triangulated

$\{F : \mathcal{C} \xrightarrow{\sim} \mathcal{C}\}$

Our nice setting: $\mathcal{C}$ = 2CY category of graph Γ.

($\Gamma = A_n$ easiest). $\bullet - \bullet - \cdots \bullet - \bullet$

$$\mathcal{C} = \langle P_1, P_2, \ldots P_n \rangle \simeq K^b(Z_n\text{-mod})/\sim$$

zigzag algebra

- $\mathrm{Hom}(P_i, P_j) = \begin{cases} 0 \\ k \\ k \oplus k \end{cases}$
- P_i spherical
- $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, A[2])^*$
- shift $[1]$

And
$P_i\langle -1 \rangle \simeq P_i[1]$

Examples

$$\left.\begin{array}{l} (P_1 \to P_2) \in \mathcal{C}. \\ P_2 \to (P_1 \to P_2) \to \mathrm{Cone}(\ ) \xrightarrow{+1} \\ \qquad\qquad\qquad\qquad\quad \cong \\ \qquad\qquad\qquad\quad P_1\langle -1\rangle[1] \simeq P_1 \end{array}\right\} \begin{array}{l} \in \text{std } \heartsuit. \\ \text{"level 0 complexes"} \end{array}$$

Features

- Each P_i induces $\sigma_{P_i} : \mathcal{C}_n \xrightarrow{\sim} \mathcal{C}_n$ [bimod $\otimes -$]
- Any spherical induces $\sigma_S : \mathcal{C}_n \xrightarrow{\sim} \mathcal{C}_n$
- $\sigma_{P_i}\sigma_{P_j} \simeq \sigma_{P_j}\sigma_{P_i}$ if $|i-j| > 1$

  $\sigma_{P_i}\sigma_{P_j}\sigma_{P_i} \simeq \sigma_{P_j}\sigma_{P_i}\sigma_{P_j}$ if $|i-j| = 1$

$\Rightarrow Br_r \circlearrowright \mathcal{C}$.

- If S & T spherical, then $\sigma_S(T)$ spherical.
- 2CY $\Rightarrow$ sphericals control most of $\mathcal{C}_n$.

E.g. $Br_r \circlearrowright \mathcal{C}_n$:

$\sigma_{P_1}(P_1) = P_1[1]$ [$P_1\langle -2\rangle[1]$]

$\sigma_{P_1}(P_2) = P_1 \to P_2$

$$\sigma_{P_1}(P_1 \to P_2) = \begin{tikzcd} & P_1 \arrow[r] & P_2 \\ P_1 \arrow[ur] \end{tikzcd}$$

$$\sigma_{P_1}\sigma_{P_3}(P_2) = \begin{tikzcd} P_1 \arrow[dr] & \\ & P_2 \\ P_3 \arrow[ur] & \end{tikzcd} \qquad \sigma_{P_1}\sigma_{P_2}(P_3) = P_1 \to P_2 \to P_3$$

$$\sigma_2\left(\begin{tikzcd} P_1 \arrow[dr] & \\ & P_2 \\ P_3 \arrow[ur] & \end{tikzcd}\right) = \begin{tikzcd} & P_1 \\ P_2 \arrow[ur] \arrow[dr] & \\ & P_3 \end{tikzcd}, \quad \sigma_2\left(\begin{tikzcd} & P_1 \\ P_2 \arrow[ur] \arrow[dr] & \\ & P_3 \end{tikzcd}\right) = \begin{tikzcd} & P_2 \arrow[r] & P_1 \\ P_2 \arrow[ur] \arrow[r] & P_2 \arrow[r] & P_3 \end{tikzcd}$$

Observation

$\sigma_S^n(X)$ eventually "attaches copies of S" to X.

[size grows linearly] $\to$ formalize later.

[Thm [BDL]: (In terms of mass, later).]

Q: More general braids? Very messy!

E.g. $(\sigma_1 \sigma_2^{-1}) \rightsquigarrow$ Fibonacci type pattern.

$$(\sigma_1 \sigma_2^{-1})(P_1) = \begin{tikzcd} & P_1 \arrow[r] & P_2 \\ P_1 \arrow[ur] \end{tikzcd}$$

$$(\sigma_1 \sigma_2^{-1})^2(P_1) = \begin{tikzcd} & & & P_2 \\ & P_1 \arrow[r] & P_2 \arrow[ur] & \\ P_1 \arrow[ur] \arrow[r] & P_2 & & \end{tikzcd}$$

size grows exponentially

$$(\sigma_1 \sigma_2^{-1})^3(P_1) = \begin{tikzcd} & & P_1 \arrow[r] & P_2 \\ P_1 \arrow[r] & P_1 \arrow[r] & P_2 \arrow[u] & \\ P_1 \arrow[r] & P_1 \arrow[r] & P_2 \arrow[u] & \end{tikzcd}$$

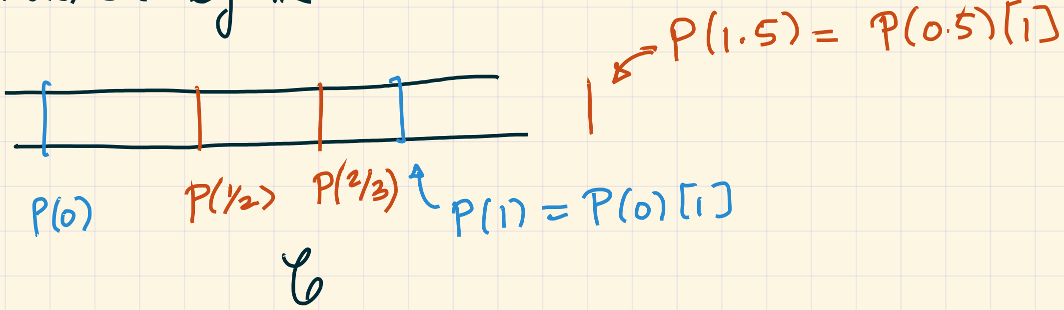
$\to$ reflects fundamental difference in σ_1 & $(\sigma_1 \sigma_2^{-1})$.

Q: How to understand growth? JH filtration very messy

* Bridgeland stability conditions

A stability condition τ on $\mathcal{C}$ prescribes:

1) A collection of additive subcategories of $\mathcal{C}$ indexed by $\mathbb{R}$



Objects of $P(\phi) :=$ semistable of phase ϕ.

2) A guarantee that $\forall\, X \in \mathcal{C}$, $\exists!$

$$\begin{tikzcd} 0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n = X \arrow[d] \\ & A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & A_n \arrow[ul, dashed] \end{tikzcd}$$

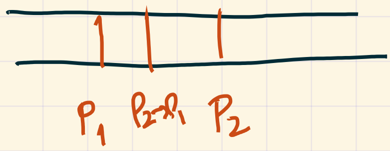
s.t $A_i \in P(\varphi_i)$ with $\varphi_1 > \varphi_2 > \cdots > \varphi_n$.

3) If $\varphi_1 > \varphi_2$ then $\mathrm{Hom}(P(\phi_1), P(\phi_2)) = 0$.

4) Each $X \in \mathcal{C}$ has a "mass" $m_\tau(X) \in \mathbb{R}_{>0}$, satisfying:

$$m_\tau(X) = \sum' m_\tau(A_i)$$

E.g.



$\mathcal{P}(\varphi) = \emptyset \;\; \forall\, \varphi \in \mathbb{R}$

bad!

Consider $(P_1 \to P_2)$: HN filtration?

$$\begin{tikzcd} 0 \arrow[rr] & & P_2 \arrow[rr] \arrow[dl] & & (P_1 \to P_2) \arrow[dl] \\ & P_2 \arrow[ul, dashed] & & P_1 \arrow[ul, dashed] & \end{tikzcd}$$

ok.

$(P_2 \to P_1)$? $\to$ No HN filtration; need to make it stable.

Now if we fix τ, we can count for any $X \in \mathcal{C}$:

1) # HN pieces of X [analogous to JH length]

2) HN <u>mass</u> of X

& measure growth!

<u>Thm</u>: $\forall\, \tau$, any spherical S:

$$m_\tau(\sigma_S^n Y) \approx m_\tau(Y) + n \cdot \dim \operatorname{Hom}(X, Y)\, m_\tau(X).$$

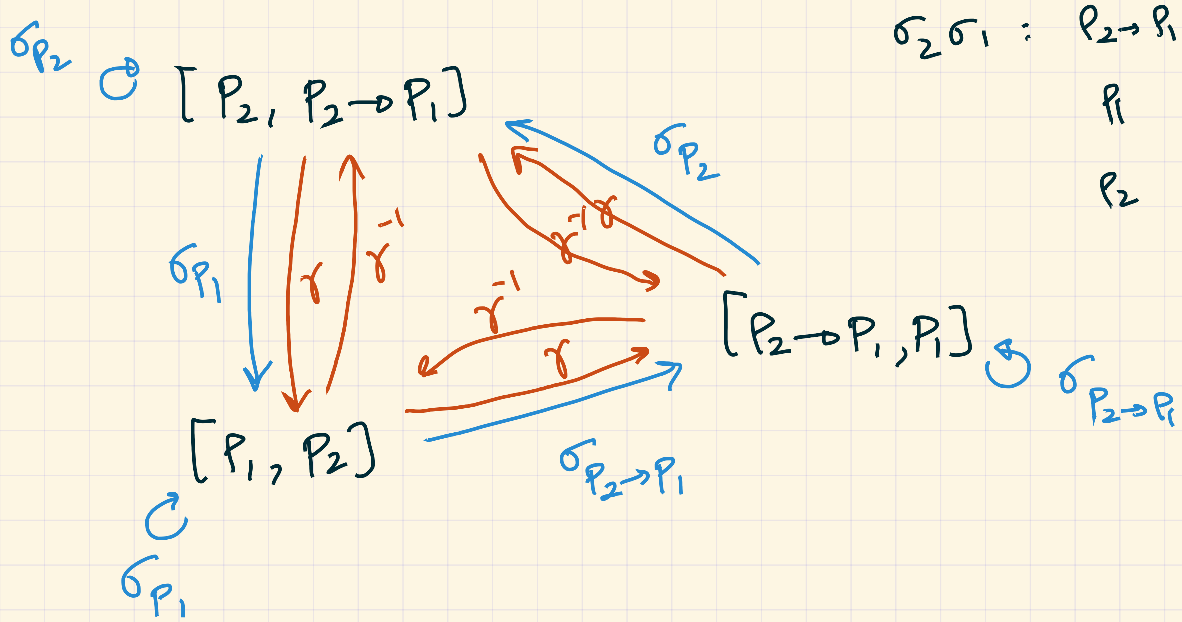
More generally, we have thms for A_2, $\hat{A}_1$, other rk 2 types [Edmund Heng].

$\to$ Let τ any stab condition, X any spherical. Then HN support of X can contain at most two of the three stable objects: $\{P_1, P_2, P_2 \to P_1\}$.

$\sigma_2\sigma_1$: $P_2 \to P_1$

P_1

P_2



<u>Thm</u>: Every arrow in the automaton changes HN multiplicities linearly.

Every object can be written as $\beta(P_1)$ or $\beta(P_2)$ where β has an expression recognised by automaton.

$\Rightarrow$ growth controlled by linear algebra!

<u>Q</u>: More general types??

<u>Converse</u>?

Given X, how much can we simplify?

<u>Thm</u>: If $X \in \mathcal{C}$ spherical & $Y =$ top HN piece, then $\sigma_Y(X)$ has lower phase spread

$\rightsquigarrow$ In type ADE, converges to phase spread ≤ 1 (in the heart) $\to$ can be used to show Stab connected.