THE SPHERE OF SPHERICAL OBJECTS

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OUTLINE

B_{n+1} $\circlearrowright$ 2CY category of type A_n

$\cup\!\mid$

B_{n+1} $\circlearrowright$ Spherical objects

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$B_{n+1} \circlearrowright$ 2CY category of type A_n + Fixed Bridgeland stability condition

$\cup\!\mid$

$B_{n+1} \circlearrowright$ Spherical objects

OUTLINE

$$
\begin{tikzcd}
B_{n+1} \circlearrowright \boxed{\text{2CY category of type } A_n} \arrow[d, phantom, "\cup"] & + \quad \boxed{\text{Fixed Bridgeland stability condition}} \arrow[d, squiggly] \\
B_{n+1} \circlearrowright \boxed{\text{Spherical objects}} \arrow[r, hook, "\text{dense}"] & \boxed{\text{PL sphere}}
\end{tikzcd}
$$

OUTLINE

$B_{n+1} \circlearrowright$ 2CY category of type A_n + Any Bridgeland stability condition $\circlearrowleft B_{n+1}$

$\cup\!\!|$

$B_{n+1} \circlearrowright$ Spherical objects $\xrightarrow{\text{dense}}$ Canonical PL sphere $\circlearrowleft$ PL B_{n+1}

THE CATEGORY & SPHERICAL OBJECTS

$$\underset{1}{\bullet} - \underset{2}{\bullet} - \cdots - \underset{n-1}{\bullet} - \underset{n}{\bullet} \qquad A_n \text{ graph}$$

$\mathcal{C}_n$ is the homotopy category of projective modules over the zigzag algebra
(a certain quotient of the path algebra of doubled quiver)

THE CATEGORY

$$\begin{tikzcd} \underset{P_1}{\bullet} \arrow[r, no head] & \underset{P_2}{\bullet} \arrow[r, no head] & \cdots & \underset{P_{n-1}}{\bullet} \arrow[r, no head] & \underset{P_n}{\bullet} \end{tikzcd} \qquad \mathcal{C}_n$$

Concretely, $\mathcal{C}_n = \langle P_1, \ldots, P_n \rangle$ with morphisms:

$$\mathrm{Hom}^{l}(P_i, P_i) = \begin{cases} \mathbb{C} & \text{if } l = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Hom}^{l}(P_i, P_j) = \begin{cases} \mathbb{C} & \text{if } l = 1 \ \& \ |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

THE CATEGORY

$$P_1 - P_2 - \cdots - P_{n-1} - P_n$$

$$\mathrm{Hom}^{\ell}(P_i, P_i) = \begin{cases} \mathbb{C} & \text{if } \ell = 0, 2 \\ 0 & \text{else} \end{cases}$$

$$\mathrm{Hom}^{\ell}(P_i, P_j) = \begin{cases} \mathbb{C} & \text{if } \ell = 1 \text{ \& } |i-j| = 1 \\ 0 & \text{else.} \end{cases}$$

Facts

- $\mathcal{C}_n$ is 2-Calabi-Yau
- Each object P_i is spherical.

The category

$$P_1 \;—\; P_2 \;—\; \cdots \;—\; P_{n-1} \;—\; P_n$$

$$\mathrm{Hom}^{\ell}(P_i, P_i) = \begin{cases} \mathbb{C} & \text{if } \ell = 0, 2 \\ 0 & \text{else} \end{cases}$$

$$\mathrm{Hom}^{\ell}(P_i, P_j) = \begin{cases} \mathbb{C} & \text{if } \ell = 1 \;\&\; |i-j| = 1 \\ 0 & \text{else.} \end{cases}$$

Facts

- $\mathcal{C}_n$ has a bounded t-structure whose heart is the extension closure of $P_1, \ldots, P_n$.

SPHERICAL OBJECTS & SPHERICAL TWISTS

An object $X \in \mathcal{C}_n$ is spherical if

$$\mathrm{Hom}^{\ell}(X,X) = \begin{cases} \mathbb{C} & \text{if } \ell = 0,2 \\ 0 & \text{else.} \end{cases}$$

Every spherical X gives rise to a twist

$$\sigma_X : \mathcal{C}_n \xrightarrow{\sim} \mathcal{C}_n.$$

SPHERICAL OBJECTS & SPHERICAL TWISTS

Recall: $P_1, P_2, \ldots, P_n$ are spherical.

Fact:
$$\sigma_{P_i}\sigma_{P_j} \cong \sigma_{P_j}\sigma_{P_i} \quad \text{if} \quad |i-j|>1$$

$$\sigma_{P_i}\sigma_{P_j}\sigma_{P_i} \cong \sigma_{P_j}\sigma_{P_i}\sigma_{P_j} \quad \text{if} \quad |i-j|=1$$

$\Rightarrow$ B_{n+1} acts on $\mathcal{C}_n$ where $\sigma_i \mapsto \sigma_{P_i}$.

DICTIONARY TO CURVES IN THE PLANE

[Khovanov – Seidel]

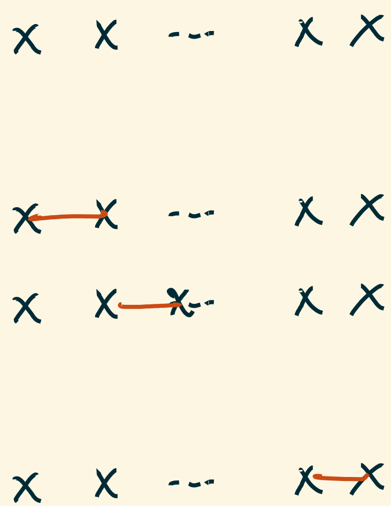
$\mathcal{C}_n$ $\qquad\qquad$ $\times \;\; \times \;\; \cdots \;\; \times \;\; \times$ $\qquad$ $(n+1)$ marked points in the plane

DICTIONARY TO CURVES IN THE PLANE

[Khovanov – Seidel]

$\mathcal{C}_n$ — (n+1) marked points in the plane

spherical generators $\left\{ \begin{array}{l} P_1 \longleftrightarrow \\ P_2 \longleftrightarrow \\ \vdots \\ P_n \longleftrightarrow \end{array} \right.$



curves connecting adjacent points.

DICTIONARY TO CURVES IN THE PLANE

[Khovanov – Seidel]

$\mathcal{C}_n$ x x ··· x x (n+1) marked points in the plane

spherical generators

$P_1 \longleftrightarrow$ x x ··· x x

$P_2 \longleftrightarrow$ x x x ··· x x

⋮

$P_n \longleftrightarrow$ x x ··· x x

curves connecting adjacent points.

$\sigma_{P_i} \longleftrightarrow$ x x ··· x x ···· x x Dehn half-twist between i & i+1

i i+1

DICTIONARY TO CURVES IN THE PLANE

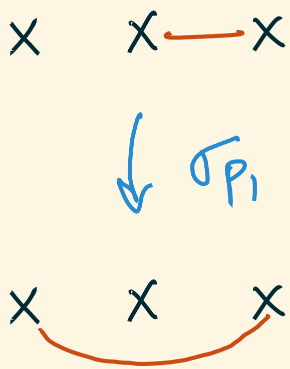
Example $(n = 2)$

P_2

DICTIONARY TO CURVES IN THE PLANE

Example $(n = 2)$

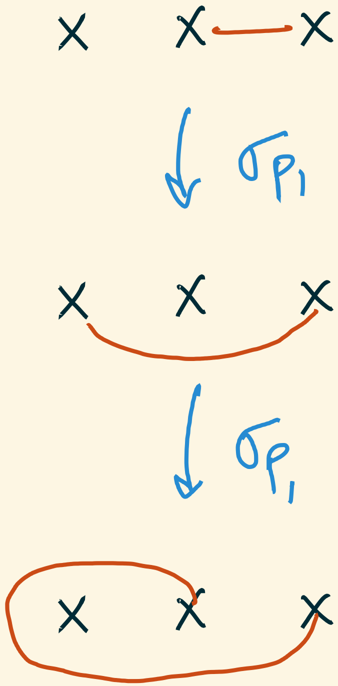
$$\begin{array}{c} P_2 \\ \downarrow \sigma_{P_1} \\ P_1 \to P_2 \end{array}$$



DICTIONARY TO CURVES IN THE PLANE

Example $(n=2)$

$$
\begin{tikzcd}
P_2 \arrow[d, "\sigma_{P_1}"] \\
P_1 \to P_2 \arrow[d, "\sigma_{P_1}"] \\
P_1 \to P_1 \to P_2
\end{tikzcd}
$$



DICTIONARY TO CURVES IN THE PLANE

[Khovanov – Seidel]

$\mathcal{C}_n$ $\qquad$ x x ... x x $\qquad$ (n+1) marked points in the plane

spherical objects $\longleftrightarrow$ non-crossing curves joining two marked pts, not passing through any other pt, up to isotopy.

DICTIONARY TO CURVES IN THE PLANE

[Khovanov – Seidel]

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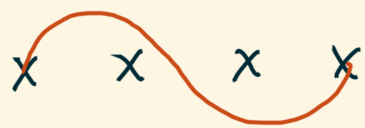
E.g. $\quad P_2 \to P_1, \; P_2 \to P_3 \quad \longleftrightarrow$

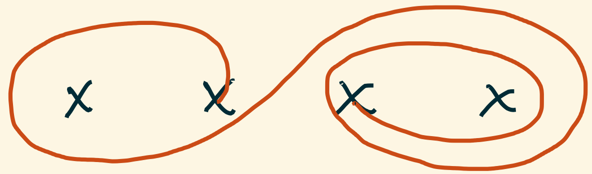


DICTIONARY TO CURVES IN THE PLANE

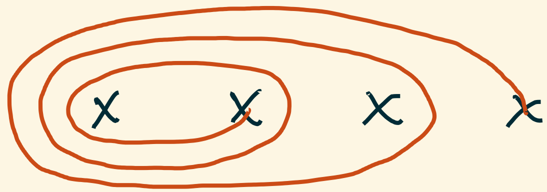
Recall: the fundamental group of the space of unordered configurations of $(n+1)$ points in the plane is B_{n+1}.

So the action of B_{n+1} on spherical objects is realised by the action of B_{n+1} on curves in a fixed configuration.

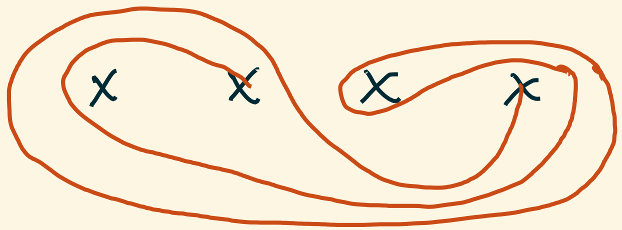
COMBINATORICS OF SPHERICALS?



<u>Q</u>: What are the constraints on the number of $P_1, \ldots, P_n$ that appear in a spherical?



<u>Q</u>: How do they evolve under the action of B_{n+1}?



BRIDGELAND STABILITY CONDITIONS ON $\mathcal{C}_n$

The data of a Bridgeland stability condition on $\mathcal{C}_n$ consists of :

1) The choice of a bounded t-structure on $\mathcal{C}_n$

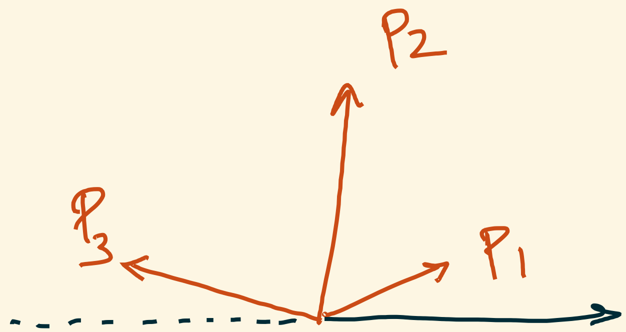
2) A stability function, i.e. a homomorphism

$$Z : K_0(\text{heart}) \longrightarrow \mathbb{H} \subseteq \mathbb{C}$$

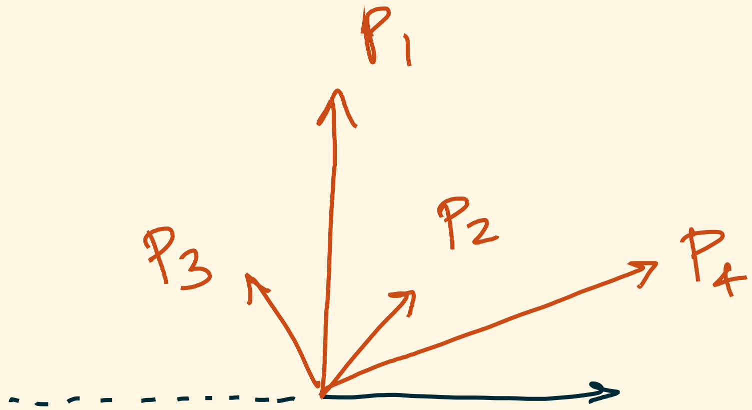
Rmk : The Harder-Narasimhan property comes for free.

BRIDGELAND STABILITY CONDITIONS ON $\mathscr{C}_n$

For simplicity, consider the standard t-structure on $\mathscr{C}_n$. A stability function is specified by a diagram as follows:



$(n=3)$



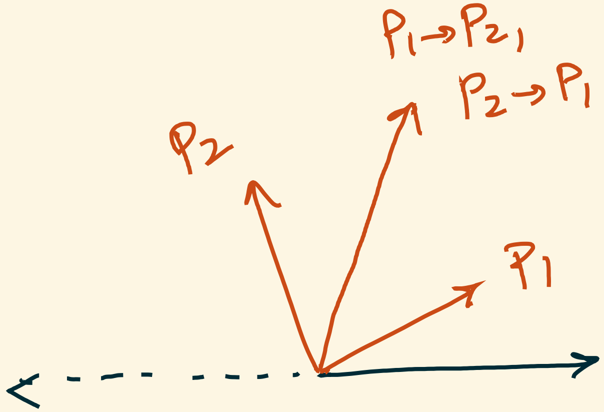
$(n=4)$

BRIDGELAND STABILITY CONDITIONS ON $\mathcal{C}_n$

Recall: Let τ be a stability condition.

$X \in \heartsuit$ is called τ-semistable if

$$\arg(Z(Y)) \leq \arg(Z(X)) \quad \forall \; 0 \neq Y \subsetneq X.$$



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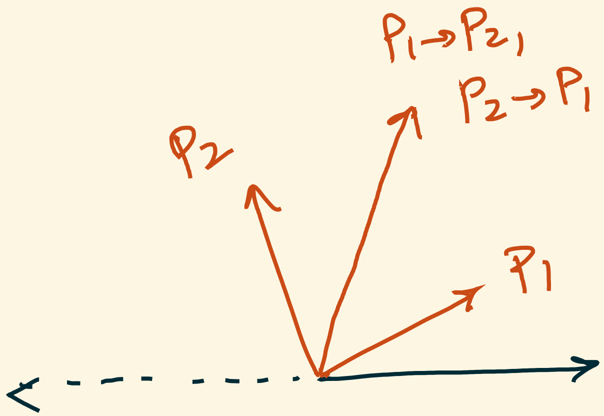

* P_1 & P_2 are simple in $\heartsuit$
$\Rightarrow$ stable

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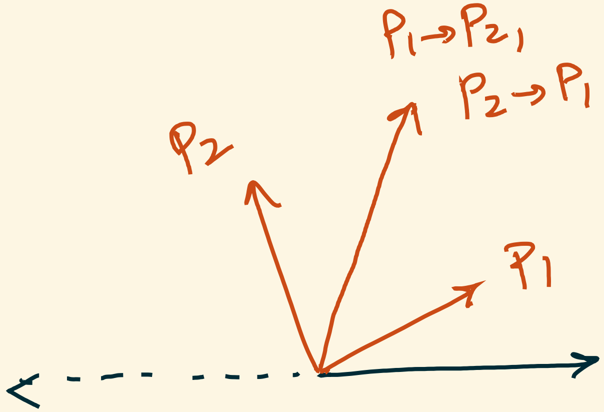
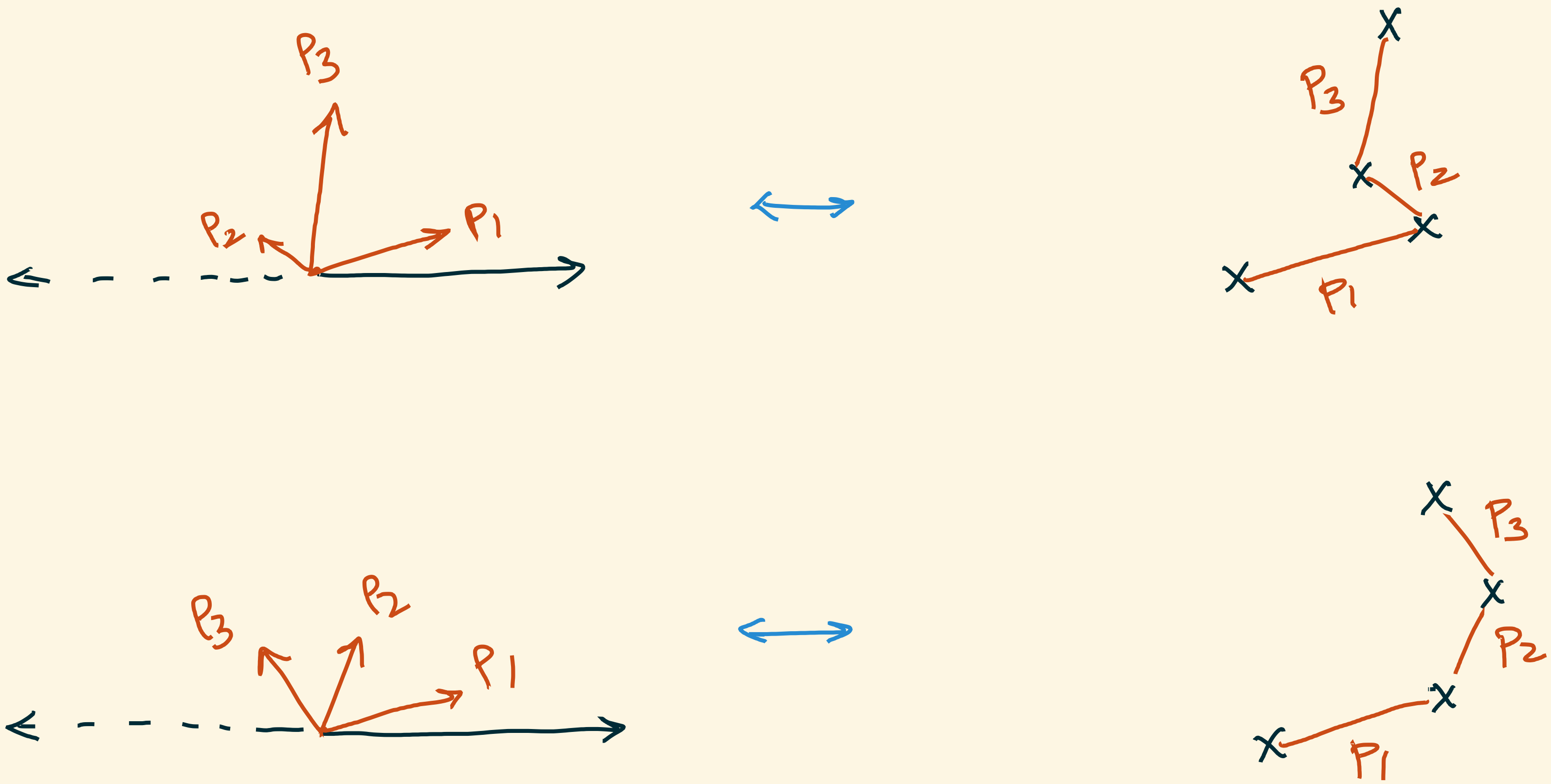


* P_1 & P_2 are simple in $\heartsuit$
  $\Rightarrow$ stable

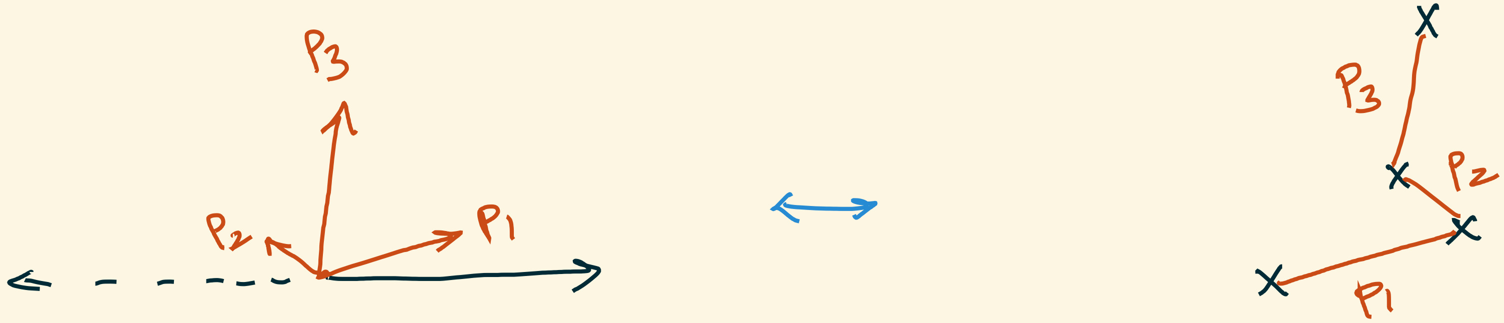
* $P_2 \subseteq (P_1 \to P_2)$ but
  $\arg(P_2) > \arg(P_1 \to P_2)$
  $\Rightarrow$ $P_1 \to P_2$ not semistable

STABILITY CONDITIONS & CURVES

Stability conditions can be expressed by varying the point configuration!

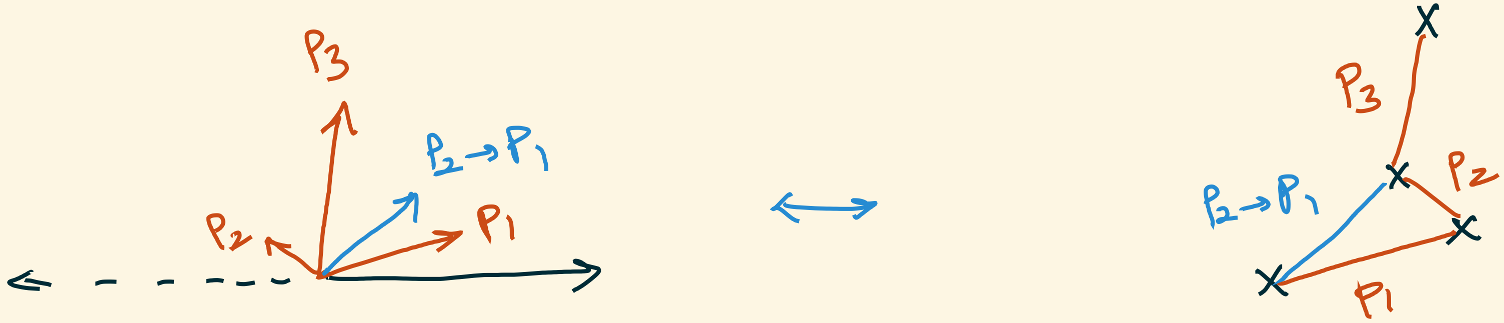


STABILITY CONDITIONS & CURVES



Observation [Thomas, BDL]: With the above recipe, semistable objects are exactly the straight-line segments between points.

STABILITY CONDITIONS & CURVES



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STABILITY CONDITIONS & CURVES

Observe: A stability function on the standard heart can be uniquely recovered via a configuration of points.

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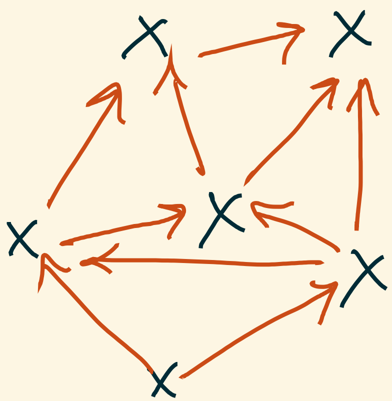
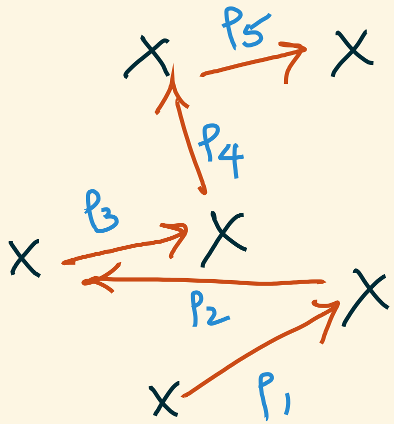
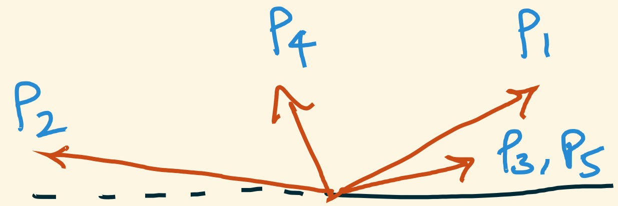






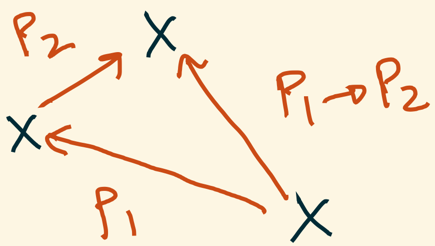
HARDER-NARASIMHAN FILTRATIONS

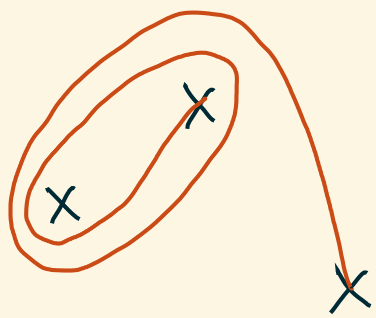
Recall: If τ is a stability condition, then any $X \in \mathcal{C}_n$ has a canonical filtration

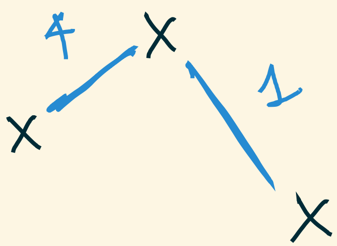
$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r, dashed] \arrow[d] & \cdots \arrow[r] & X_k = X \arrow[d] \\
& A_1 & A_2 \arrow[ul, dotted] & & A_k \arrow[ul, dotted]
\end{tikzcd}$$

such that $A_1, A_2, \cdots, A_k$ are τ-semistable, and $A_1, \cdots A_k$ have decreasing phase.

H-N FILTRATIONS VIA CURVES



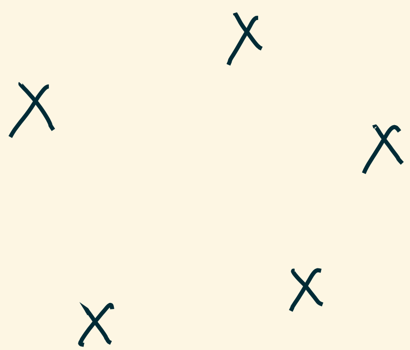




Prop [BDL]: If X is spherical, its HN filtration pieces are found by pulling the curve tight around the punctures & counting multiplicity.

COORDINATES ON SPHERICAL OBJECTS

For simplicity, fix a "convex" stability condition τ.



COORDINATES ON SPHERICAL OBJECTS

Observe : The HN support of any spherical always lies on a triangulation \ external edge.

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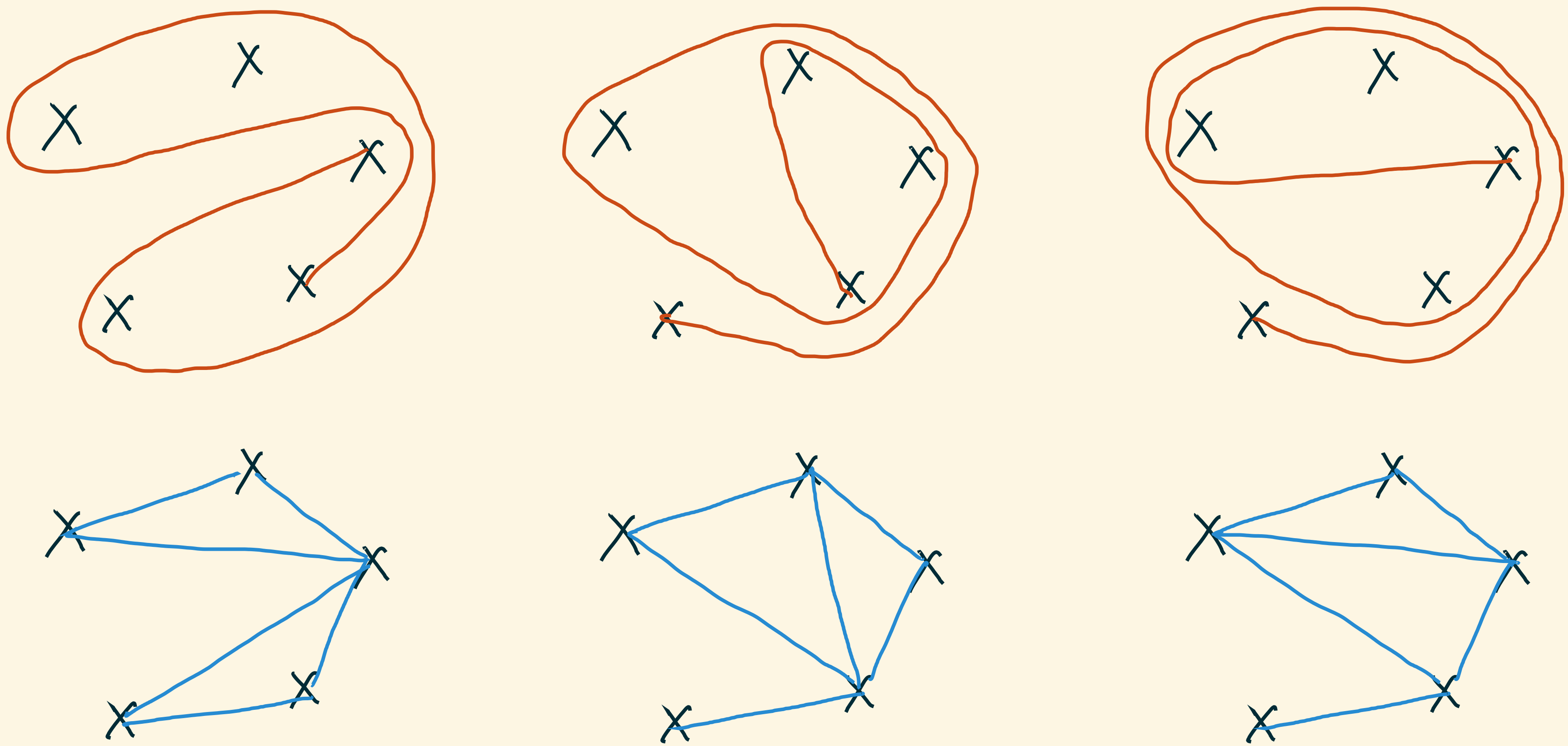
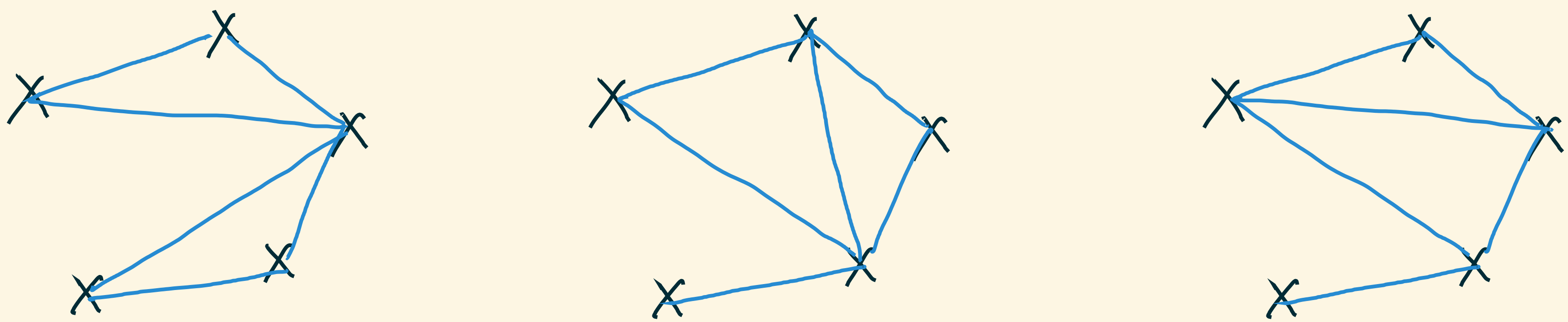


COORDINATES ON SPHERICAL OBJECTS

Observe: The HN support of any spherical always lies on a triangulation \ external edge.

$\Rightarrow$ The HN support has at most $(2n-2)$ distinct semistable pieces.



COORDINATES ON SPHERICAL OBJECTS

We have a map

$$\{\text{sphericals}\} \longrightarrow \mathbb{R}^{\binom{n+1}{2}}, \quad \text{sending}$$

$$X \longmapsto \text{HN multiplicity vector.}$$

The image of any spherical X has at most $(2n-2)$ nonzero coordinates.

COORDINATES ON SPHERICAL OBJECTS

In fact, the map

$$\{\text{sphericals}\} \longrightarrow \mathbb{R}^{\binom{n+1}{2}}$$

is injective: a spherical object can be recovered from its HN multiplicity vector.

COORDINATES ON SPHERICAL OBJECTS

Consider the projectivised map

$$\{\text{sphericals}\} \longrightarrow \mathbb{PR}^{\binom{n+1}{2}}$$

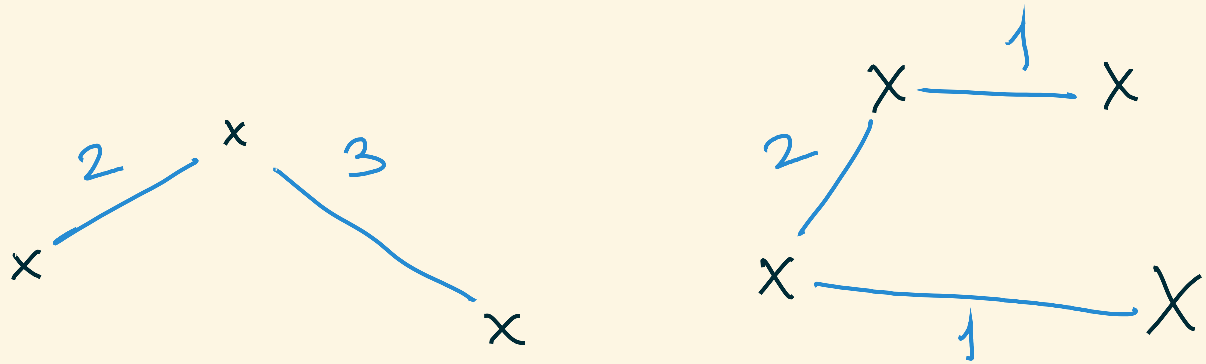
Facts: Up to scaling:

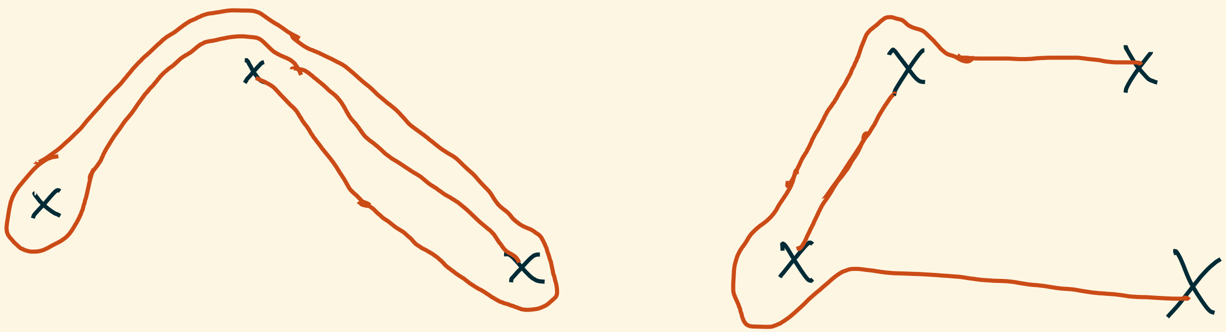
* We can reconstruct a multi-curve from any positive integer coordinates with the correct support

* Multicurves $\approx$ curves.

COORDINATES ON SPHERICAL OBJECTS

Examples





COORDINATES ON SPHERICAL OBJECTS

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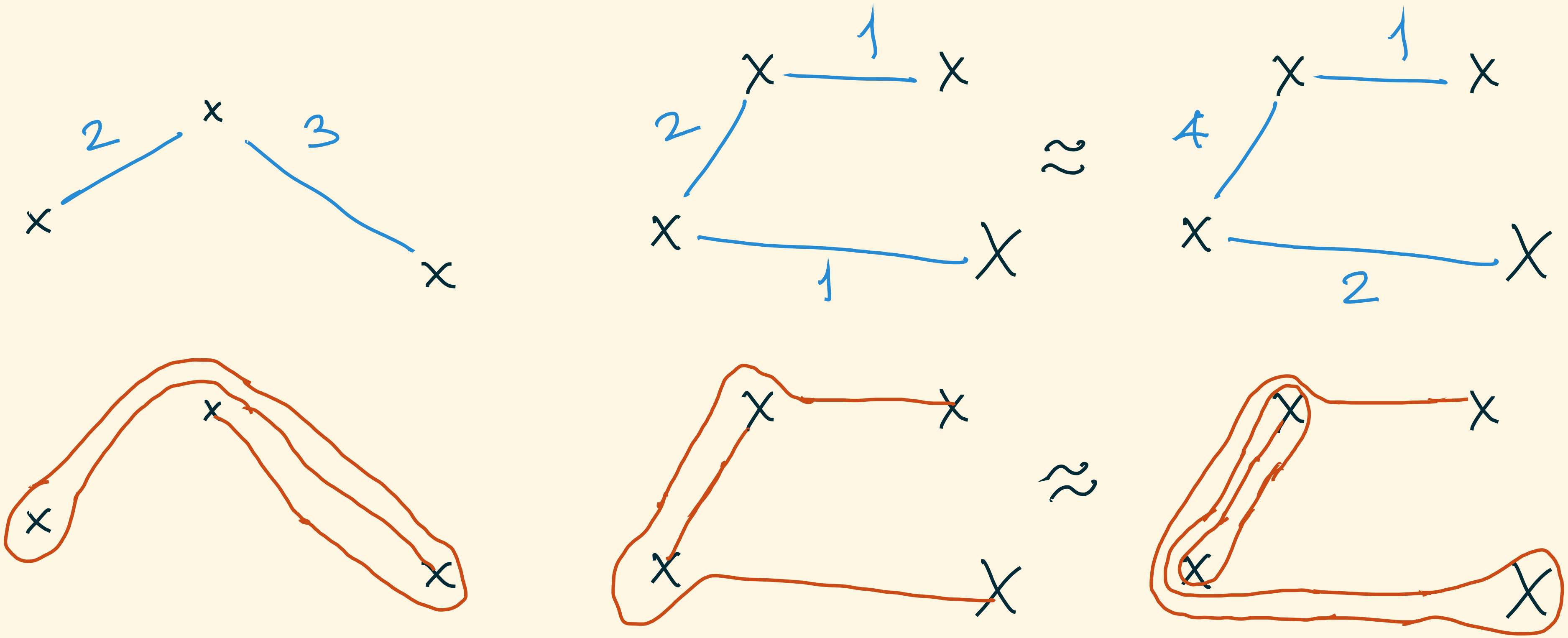
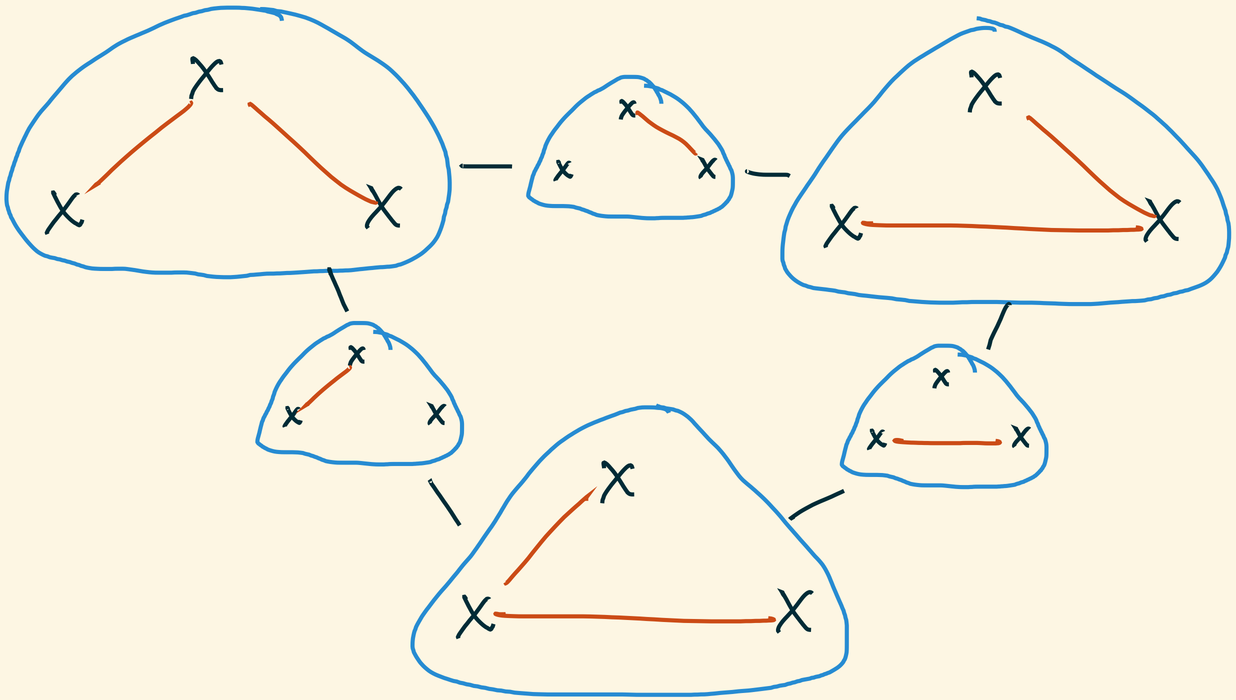


COORDINATES ON SPHERICAL OBJECTS

Consider the simplicial complex whose maximal simplices are triangulations minus an external edge.



COORDINATES ON SPHERICAL OBJECTS

Consider the simplicial complex whose maximal simplices are triangulations minus an external edge.

Proposition

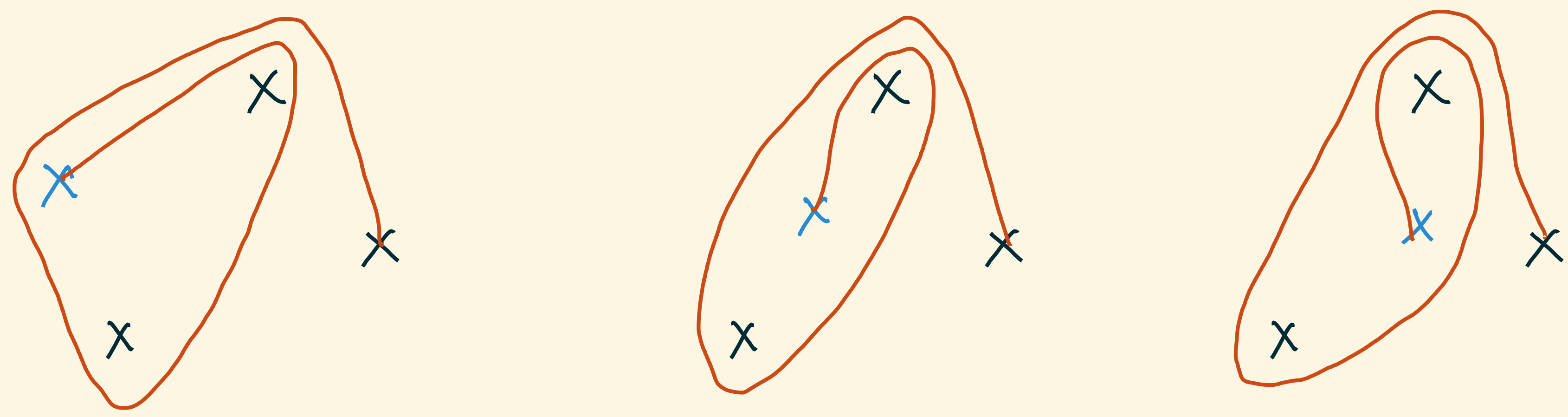
Its geometric realisation is a sphere of dimension $2n-3$.

THE SPHERE OF SPHERICALS

Result:

* We have found a simplicial complex that is homeomorphic to a sphere.
* The spherical objects of $\mathcal{C}_n$ form a dense subset of this sphere.

OTHER STABILITY CONDITIONS

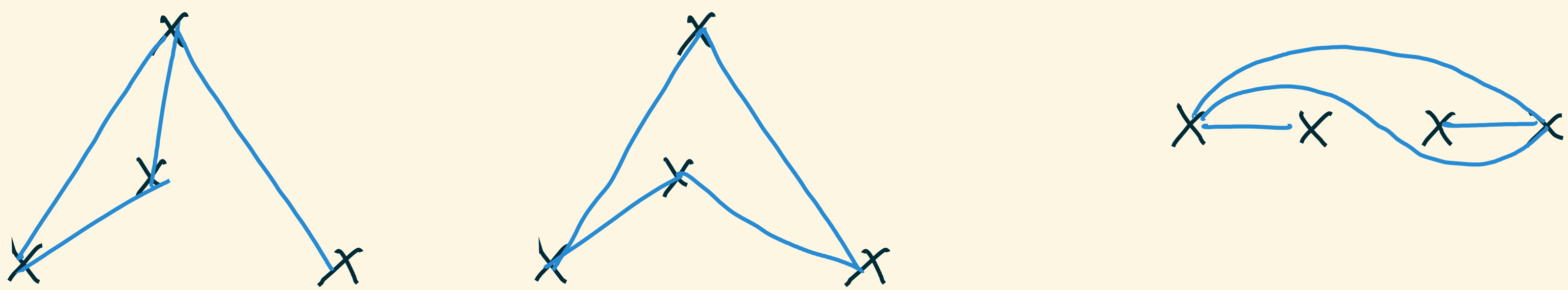


The coordinates change after wall-crossing to a new stability condition.

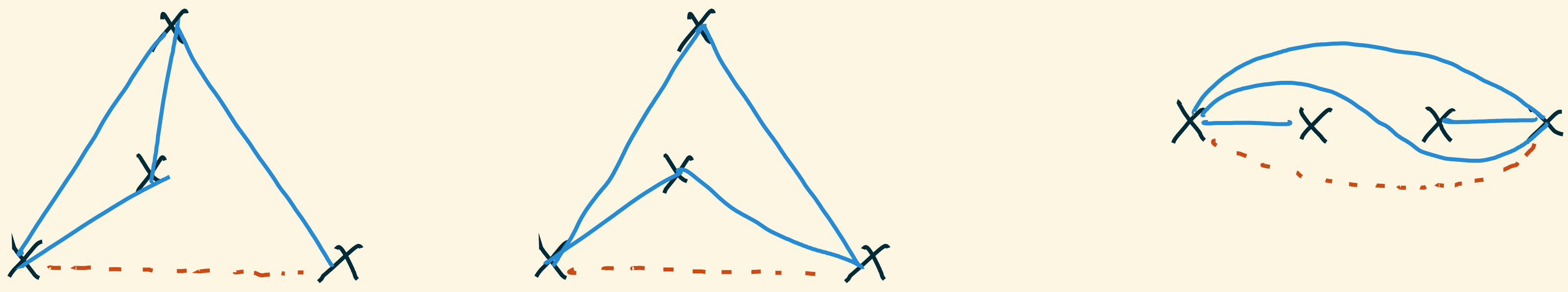
Q: Behaviour for other stability conditions?

OTHER STABILITY CONDITIONS

Observation [BDL] : For a general configuration of points, the support of a curve always lies on a "pointed pseudo-triangulation" minus an external edge.



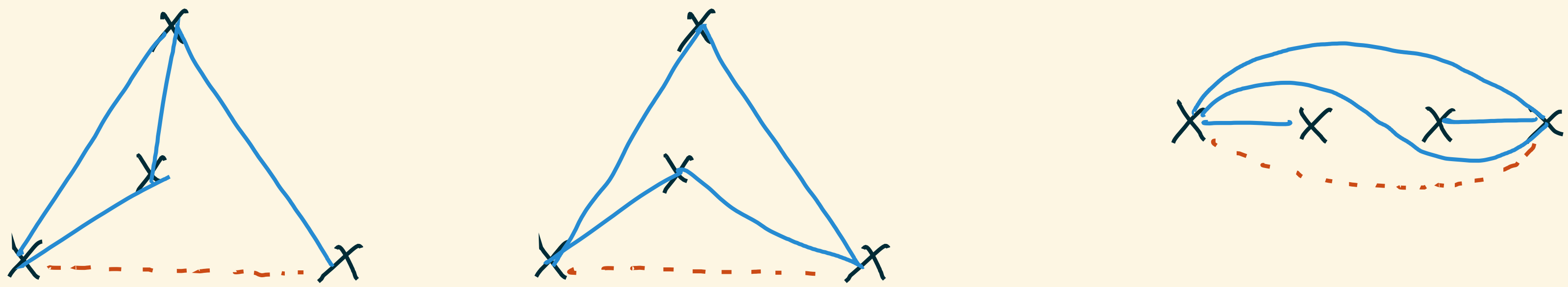
OTHER STABILITY CONDITIONS



Pointed pseudo-triangulation (ppt) is a maximal collection of edges such that

* No two cross
* Every vertex has a reflex angle.

OTHER STABILITY CONDITIONS



Facts

* Every ppt minus an external edge has $(2n-2)$ edges.

* For a fixed configuration, ppts minus an external edge form a piecewise-linear sphere.

THE (CANONICAL) SPHERE OF SPHERICALS

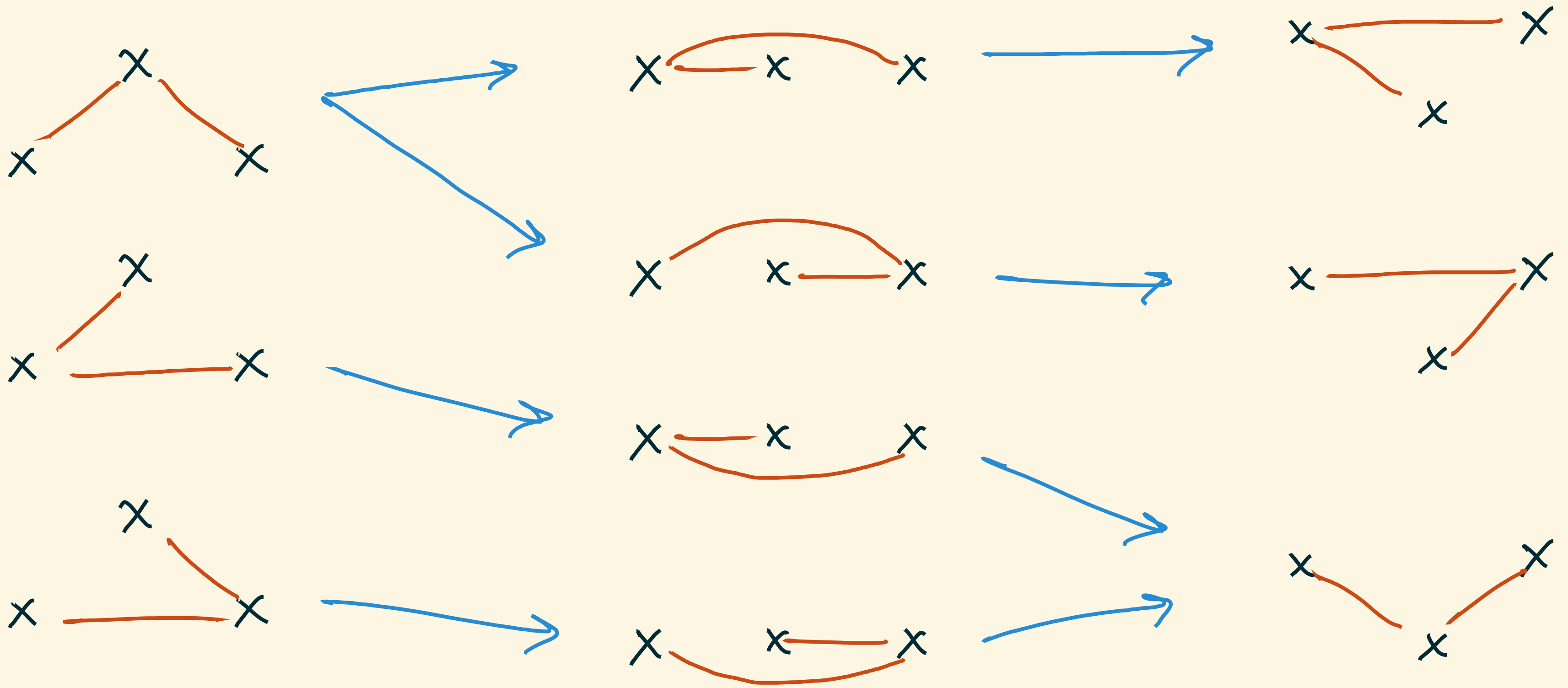
Thm [BDL]: Let τ be any stability condition on the 2CY category of the A_n quiver.

① Associated to τ is a PL sphere of dim $(2n-3)$, on which the spherical objects are dense.

② As τ changes by wall-crossing, the PL spheres transform via PL homeomorphisms.

THE (CANONICAL) SPHERE OF SPHERICALS

Example



THE (CANONICAL) SPHERE OF SPHERICALS

Corollary: There is a canonical PL "sphere of sphericals" associated to $\mathcal{C}_n$.

Thm [BDL]: The action of B_{n+1} on this sphere is piecewise-linear.

WORK IN PROGRESS

* The sphere of sphericals $\longleftrightarrow$ boundary of the space of stability conditions

* Types other than A_n.

Q: Is the group of PL automorphisms of this sphere just the braid group?

THANKS!