Bridgeland stability conditions, spherical objects, and autoequivalences

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Overview

* Setting: triangulated category with a large group of autoequivalences

* Motivating question:
  - Can we control how objects evolve under auto-equivalences?

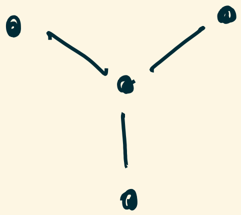
Overview

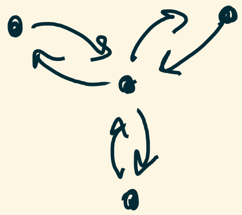
* **Main tools:**
  - Bridgeland stability conditions
  - Spherical objects.

* **Outputs:**
  - Control over how objects evolve
  - Algorithms to simplify complicated objects.

Setting

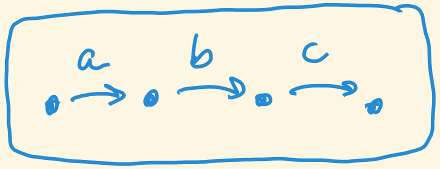
Γ a graph & $\overline{\Gamma}$ the doubled quiver



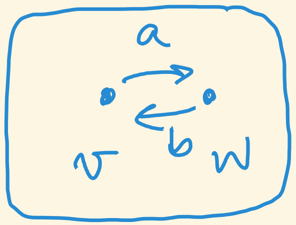


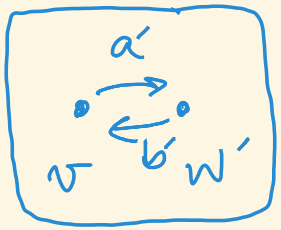
A_Γ = zigzag algebra = $k\overline{\Gamma}/\sim$

Relations :



$abc = 0$





$ab = a'b'$

The associated category

Γ a graph & $\overline{\Gamma}$ the doubled quiver

A_Γ = zigzag algebra = $k\overline{\Gamma}/\sim$

$$\mathcal{C}_\Gamma = K^b(A_\Gamma\text{-proj}) \Big/ \text{grading collapse}$$

The associated category

Γ a graph & $\overline{\Gamma}$ the doubled quiver

$$A_\Gamma = \text{zigzag algebra} = k\overline{\Gamma}/\sim$$

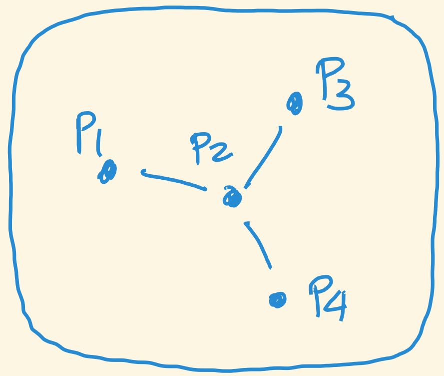
$$\mathcal{C}_\Gamma = K^b(A_\Gamma\text{-proj}) \Big/ \text{grading collapse}$$

$$\underset{\text{internal}}{X\langle -1\rangle} \cong X\underset{\text{homological}}{\{1\}}$$

The associated category

$$\mathcal{C}_r = K^b(A_r\text{-proj}) \Big/ \text{grading collapse}$$

Features of $\mathcal{C}_r$:



- Generated by $P_1, P_2, \ldots, P_n$
- Triangulated (has shift $[1]$)
- Has a "standard" t-structure whose heart is generated by the P_i

Spherical objects and spherical twists

- $\mathrm{Hom}^n(P_i, P_j) = \begin{cases} k & \text{if } i = j \ \& \ n = 0, 2 \\ k & \text{if } i - j \ \& \ n = 1 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow$ Each P_i is spherical

- Any spherical object S induces

$$\sigma_S : \mathcal{C}_\Gamma \xrightarrow{\sim} \mathcal{C}_\Gamma , \quad \text{where}$$

$$\sigma_S(X) := \mathrm{Cone}\left(\mathrm{Hom}(S, X) \otimes S \xrightarrow{ev} X\right)$$

Spherical objects and spherical twists

In particular, we have

$\sigma_{P_i} : \mathcal{C}_\Gamma \longrightarrow \mathcal{C}_\Gamma$, which satisfy

$$\sigma_{P_i}\sigma_{P_j}\sigma_{P_i} = \sigma_{P_j}\sigma_{P_i}\sigma_{P_j} \quad \text{if} \quad i - j$$

$$\sigma_{P_i}\sigma_{P_j} = \sigma_{P_j}\sigma_{P_i} \quad \text{if} \quad i \not- j$$

$\Rightarrow B_\Gamma$ acts on $\mathcal{C}_\Gamma$ by autoequivalences.

Spherical objects and spherical twists

Example:

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, no head] & \underset{2}{\bullet} \arrow[r, no head] & \underset{3}{\bullet} \end{tikzcd}$$

$$\sigma_{P_1}(P_1) = \mathrm{Cone}\left(\mathrm{Hom}(P_1, P_1) \otimes P_1 \xrightarrow{ev} P_1\right)$$

$$= \mathrm{Cone}\left(P_1\langle -2\rangle \oplus P_1 \longrightarrow P_1\right)$$

$$= P_1\langle -2\rangle\{1\}$$

$$\sigma_{P_1}(P_1) = P_1[1] \qquad \text{(by grading collapse)}$$

Spherical objects and spherical twists

Example:

$$\boxed{\underset{1}{\bullet} \;-\!\!-\!\!-\; \underset{2}{\bullet} \;-\!\!-\!\!-\; \underset{3}{\bullet}}$$

$$\sigma_{P_1}(P_2) = \operatorname{Cone}\left(\operatorname{Hom}(P_1, P_2) \otimes P_1 \xrightarrow{ev} P_2 \right)$$

$$= \operatorname{Cone}\left(P_1\langle -1 \rangle \longrightarrow P_2 \right)$$

$$\sigma_{P_1}(P_2) = P_1 \longrightarrow P_2$$

Spherical objects and spherical twists

Example: $\boxed{\underset{1}{\bullet} \;\text{---}\; \underset{2}{\bullet} \;\text{---}\; \underset{3}{\bullet}}$

$$\sigma_{P_1}\sigma_{P_3}(P_2) = \begin{tikzcd}[row sep=small] P_1 \arrow[dr] & \\ \oplus & P_2 \\ P_3 \arrow[ur] & \end{tikzcd} \quad , \quad \sigma_{P_1}\sigma_{P_2}(P_3) = P_1 \to P_2 \to P_3$$

$$\sigma_2\left(\begin{tikzcd}[row sep=small] P_1 \arrow[dr] & \\ \oplus & P_2 \\ P_3 \arrow[ur] & \end{tikzcd}\right) = \begin{tikzcd}[row sep=small] & P_1 \\ P_2 \arrow[ur] \arrow[dr] & \\ & P_3 \end{tikzcd} \quad , \quad \sigma_2(P_1 \to P_2 \to P_3) = P_1 \to P_2 \to P_3.$$

Observations

* $\sigma_S^n(X)$ eventually "attaches copies of S" to X.

Example : $\sigma_{P_1}(P_2) = P_1 \to P_2$

$\sigma_{P_1}^2(P_2) = P_1 \to P_1 \to P_2$

$\sigma_{P_1}^3(P_2) = P_1 \to P_1 \to P_1 \to P_2$

$\vdots$

→ Linear growth

Observations

* $\sigma_s^n(x)$ eventually "attaches copies of s" to x.

* Other braids are much more complicated

Example : $\beta = \sigma_{P_1} \sigma_{P_2}^{-1}$

$$\beta(P_1) = \begin{tikzcd} & P_1 \arrow[r] & P_2 \\ P_1 \arrow[ur] & & \end{tikzcd}$$

$$\beta^2(P_1) = \begin{tikzcd} & & & P_2 \\ & P_1 \arrow[r] & P_2 \arrow[ur] & \\ P_1 \arrow[r] \arrow[ur] & P_2 & & \end{tikzcd}$$

$$\beta^3(P_1) = \begin{tikzcd} & P_1 \arrow[r] & P_2 \\ & P_1 \arrow[r] & P_2 \arrow[u] \\ P_1 \arrow[ur] & & \\ & P_1 \arrow[r] & P_2 \arrow[uu] \\ P_1 \arrow[ur] & & \end{tikzcd}$$

$\rightarrow$ Exponential growth.

Questions

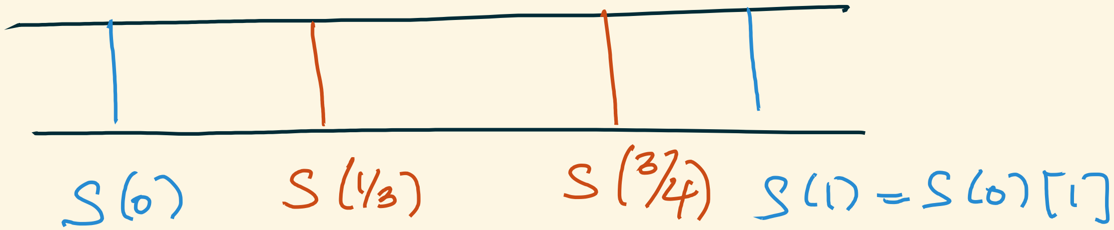
- How to understand growth?

- How can we
  (a) simplify spherical objects, and
  (b) construct desired ones algorithmically?

Bridgeland stability conditions

A stability condition prescribes:

1) A collection of additive subcategories of $\mathcal{C}$ indexed by $\mathbb{R}$, compatible with shifts.



[This is called the *slicing*.]

Bridgeland stability conditions

A stability condition prescribes:

1) A slicing $\{S(\phi) \mid \phi \in \mathbb{R}\}$,

2) $\forall\, X \in \mathcal{C}$, a unique filtration

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n = X \arrow[d] \\
& A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & A_n \arrow[ul, dashed]
\end{tikzcd}$$

with $A_i \in S(\phi_i)$ &

$\phi_1 > \phi_2 > \cdots > \phi_n$.

[This is called the Harder-Narasimhan filtration.]

Bridgeland stability conditions

A stability condition prescribes:

1) A slicing $\{\mathcal{S}(\phi) \mid \phi \in \mathbb{R}\}$,

2) A unique HN filtration for each $X \in \mathcal{C}_r$

3) A mass $m(A) \in \mathbb{R}_{>0}$ for each $A \in \mathcal{S}(\phi)$.

[If $X \notin$ any $\mathcal{S}(\phi)$ then

$m(X) := \sum' m(A_i)$ over the HN pieces.]

Bridgeland stability conditions

A stability condition prescribes:

1) A slicing $\{S(\phi) \mid \phi \in \mathbb{R}\}$,

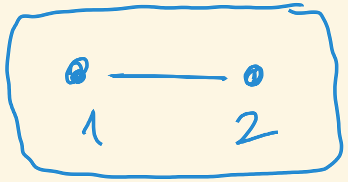
2) A unique HN filtration for each $X \in \mathcal{C}_{\Gamma}$

3) A mass $m(A) \in \mathbb{R}_{>0}$ for each $A \in S(\phi)$.

+ compatibility conditions.

Bridgeland stability conditions

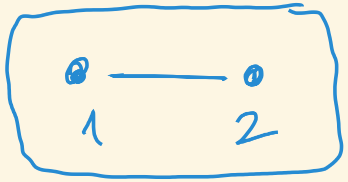
Example: Type A_2

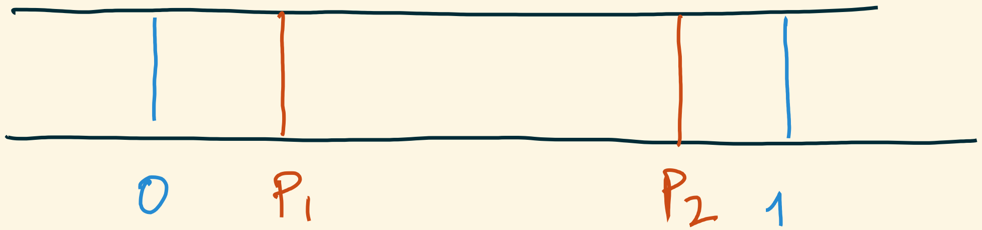


Bridgeland stability conditions

Non-

Example : Type A_2





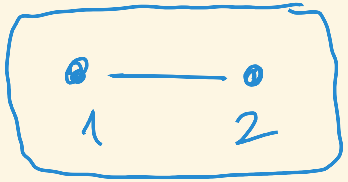
$(P_1 \to P_2)$ has HN filtration

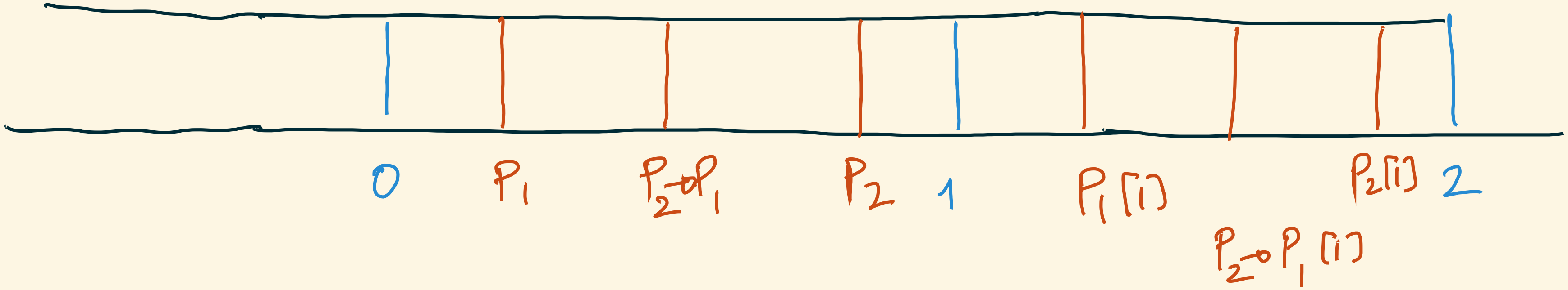
$$
\begin{tikzcd}
0 \arrow[rr] & & P_2 \arrow[rr] \arrow[dl] & & (P_1 \to P_2) \arrow[dl] \\
& P_2 \arrow[ul, dotted] & & P_1 \arrow[ul, dotted] &
\end{tikzcd}
$$

but $P_2 \to P_1$ does not.

Bridgeland stability conditions

Example : Type A_2

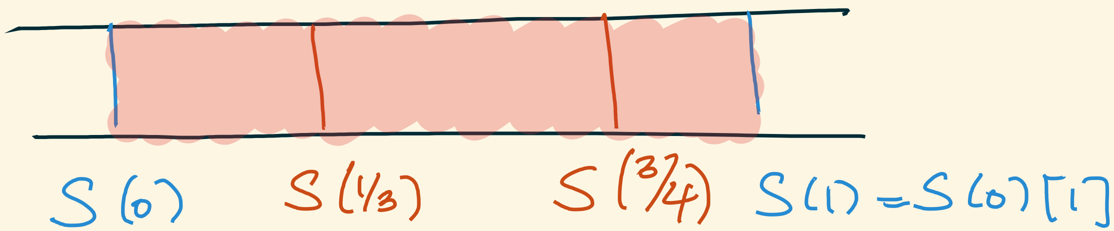




This slicing gives a stability condition.

Bridgeland stability conditions

- A stability condition is said to be <u>standard</u> if $S([0,1))$ is the standard heart.



- In this case, the HN filtration is a reorganisation of the cohomology / JH filtration.

Growth via stability conditions

If we fix a stability condition τ, we can count for any $X \in \mathcal{C}$, e.g.:

1) the number of HN pieces of X, or

2) the HN mass of X, or

3) the "phase spread" of X.

Growth via stability conditions

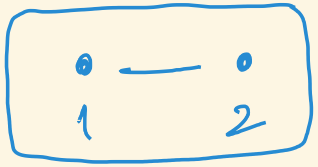
Consider $\mathcal{C}_\Gamma$ as before.

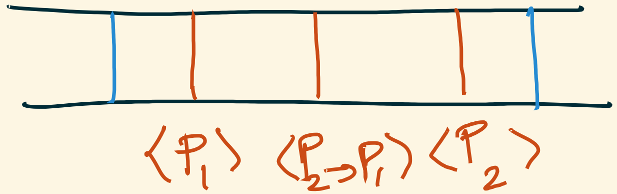
Theorem [B - Deopurkar - Licata]

For any stability condition τ and any spherical object X, we have:

$$m_\tau(\sigma_s^n X) \approx m_\tau(X) + n \cdot [\dim \mathrm{Hom}^*(S, X)] \cdot m_\tau(S)$$

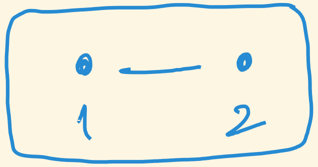
Growth via stability conditions

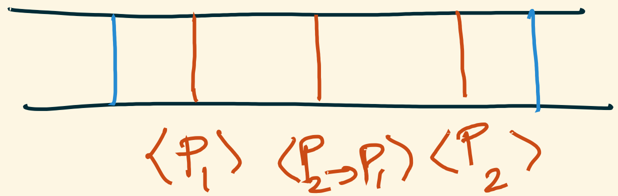
Type A_2 

, fix τ: 

In this case, we can say much more!

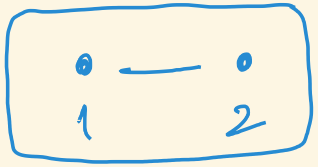
Growth via stability conditions

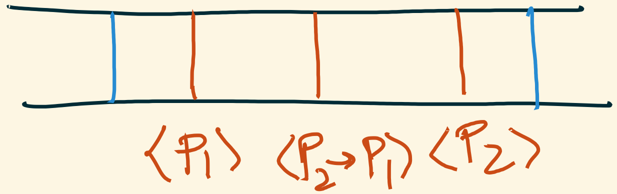
Type A_2 

, fix τ: 

Prop [BDL]: If X spherical, then the HN support contains at most two out of the three objects $\{P_1, P_2 \to P_1, P_2\}$, up to shift.

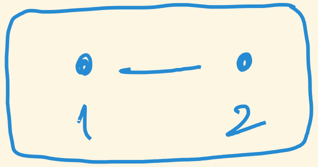
Growth via stability conditions

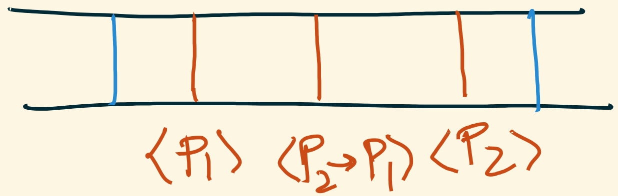
Type A_2 

, fix τ: 

Thm [BDL] The action of the braid group is controlled by a finite automaton.

Growth via stability conditions

Type A_2 

, fix τ: 

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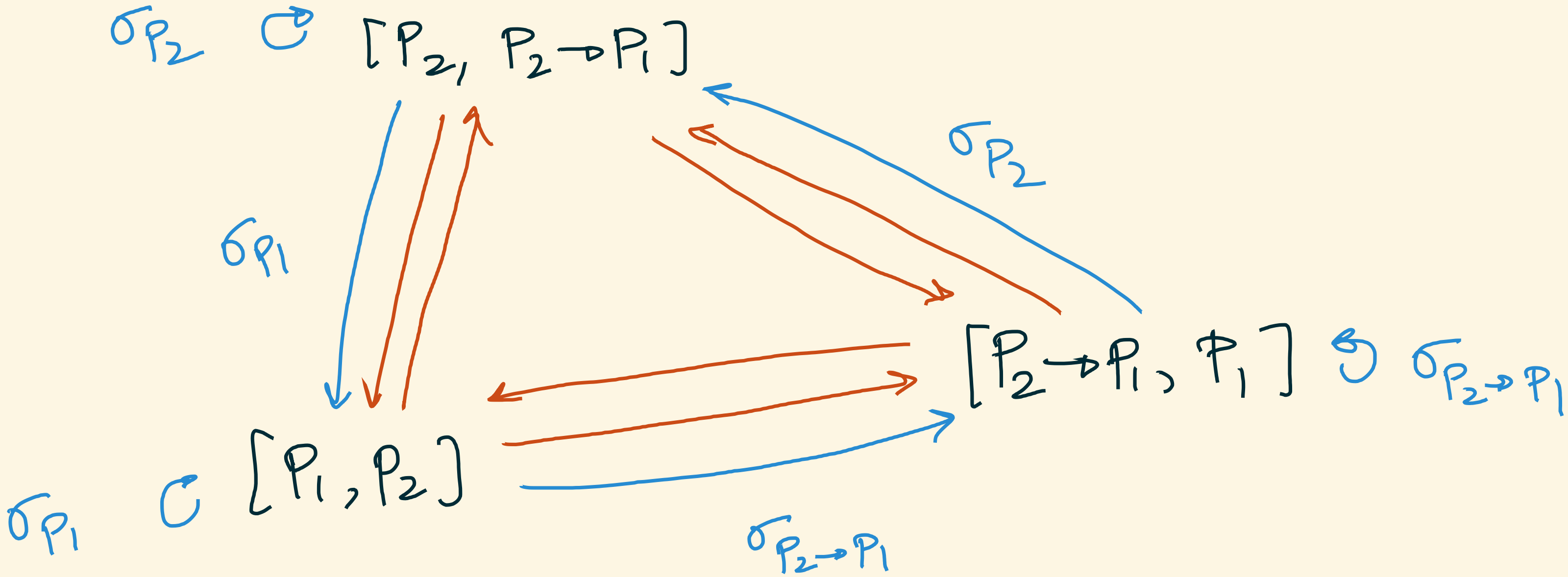
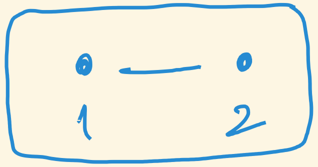
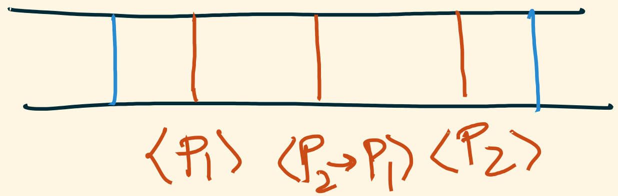


Growth via stability conditions

Type A_2 

, fix τ: 

Thm [BDL] The action of the braid group is controlled by a finite automaton:

1) every arrow changes HN multiplicities linearly;

2) every spherical object can be written as $\beta(P_1)$ or $\beta(P_2)$ via recognised expressions

Growth via stability conditions

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
{[P_2, P_2 \to P_1]} \arrow[loop left, "\sigma_{P_2}"] \arrow[d, shift right=3, "\sigma_{P_1}"'] \arrow[d, shift right=1] \arrow[dr, shift left, "id"] & \\
{[P_1, P_2]} \arrow[loop left, "\sigma_{P_1}"] \arrow[u, shift right=1, "id"'] \arrow[r, shift left=1] \arrow[r, shift right=3, "\sigma_{P_2 \to P_1}"'] & {[P_2 \to P_1, P_1]} \arrow[loop right, "\sigma_{P_2 \to P_1}"] \arrow[l, shift left=1, "id"'] \arrow[ul, shift left, "id"] \arrow[ul, shift right=3, "\sigma_{P_2}"']
\end{tikzcd}
$$

$$\sigma_{P_1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \sigma_{P_2 \to P_1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow$ Growth controlled by linear algebra!

Growth via stability conditions

We can write down automata for $\hat{A}_1$ and

- other rank-2 cases [E. Heng]
- a q-analogue for A_2 [w/ L. Becker, A. Licata]

Question: Automata / finer growth control for other types?

Simplification via stability conditions

Consider any $\mathcal{C}_\Gamma$, and fix a stability condition τ.

Recall the HN filtration:

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] & & X_1 \arrow[rr] \arrow[dl] & & X_2 \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] & & X_n \arrow[rr] \arrow[dl] & & X \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] & & &
\end{tikzcd}$$

<u>Fact</u>: If X spherical, then each stable piece is spherical.

Simplification via stability conditions

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& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] & & &
\end{tikzcd}$$

Fact: If X spherical, then each stable piece is spherical.

Recall: The phase spread of X is

$$\phi(A_1) - \phi(A_n).$$

Simplification via stability conditions

Consider any $\mathscr{C}_\Gamma$, and fix a stability condition τ.

Simplification via stability conditions

Consider any $\mathcal{C}_\Gamma$, and fix a stability condition τ.

Thm [BDL]

Let X be spherical. Let Y be the HN piece of X of highest phase. Suppose Y is spherical. If the phase spread of X is > 0, then the phase spread of $\sigma_Y(X)$ is lower than the phase spread of X.

Simplification via stability conditions

Thm (summary) If Y = top HN piece of X,

phase spread $(\sigma_Y(X))$ < phase spread (X).

In finite type (ADE), this gives an algorithm that converges to phase spread = 0.

(i.e. a stable object)

Simplification via stability conditions

Thm (summary) If Y = top HN piece of X,

phase spread $(\sigma_Y(X))$ < phase spread (X).

Cor (ADE type)

1) All spherical objects lie in one orbit of the braid group

2) The space of stability conditions is connected.

Simplification via stability conditions

Thm (summary) If Y = top HN piece of X,

$$\text{phase spread}(\sigma_Y(X)) < \text{phase spread}(X).$$

Cor (ADE type)

1) All spherical objects lie in one orbit of the braid group

2) The space of stability conditions is connected.

(originally Ishii-Ueda-Uehara, Ishii-Uehara, Adachi-Mizuno-Yang, ...)

Simplification via stability conditions

Example: Type A_2 $\quad \boxed{\underset{1}{\circ} \text{———} \underset{2}{\circ}}$, stables

$$P_1$$
$$P_2 \to P_1$$
$$P_2$$

$$X = \quad \begin{array}{l} \qquad\qquad (P_2 \to P_1)[2] \\ \qquad\quad \nearrow \\ \qquad P_2[1] \\ \quad \nearrow \\ P_2 \end{array}$$

Simplification via stability conditions

Example: Type A_2 $\boxed{\underset{1}{\bullet} \text{———} \underset{2}{\bullet}}$, stables

$$P_1$$
$$P_2 \to P_1$$
$$P_2$$

$$X = \begin{array}{lll} & & \underline{(P_2 \to P_1)[2]} \\ & \nearrow & \\ & \underline{P_2[1]} & \\ \nearrow & & \\ \underline{P_2} & & \end{array} \quad , \quad Y_1 = (P_2 \to P_1)[2]$$

Simplification via stability conditions

Example: Type A_2 $\quad \boxed{\underset{1}{\bullet} \text{———} \underset{2}{\bullet}}$, stables

$$P_1$$
$$P_2 \to P_1$$
$$P_2$$

$$X = \begin{tikzcd}[column sep=small, row sep=small] & & (P_2 \to P_1)[2] \\ & P_2[1] \arrow[ur] & \\ P_2 \arrow[ur] & & \end{tikzcd} \quad , \quad Y_1 = (P_2 \to P_1)[2]$$

$$\sigma_{Y_1}(X) = P_1 \to P_2$$

Simplification via stability conditions

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Simplification via stability conditions

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$$X = \begin{tikzcd}[column sep=small, row sep=small] & & (P_2 \to P_1)[2] \\ & P_2[1] \arrow[ur] & \\ P_2 \arrow[ur] & & \end{tikzcd} \quad , \quad Y_1 = (P_2 \to P_1)[2]$$

$$\sigma_{Y_1}(X) = P_1 \to P_2 \quad , \quad Y_2 = P_2$$

$$\sigma_{Y_2}\sigma_{Y_1}(X) = P_1 \quad , \quad \text{hence} \quad X = \sigma_{Y_1}^{-1}\sigma_{Y_2}^{-1}(P_1)$$

Stability conditions $\to$ braid words

Example: $X = $

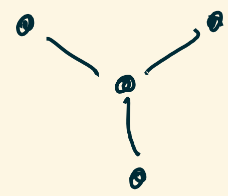
$$\begin{array}{l} \qquad\qquad (P_2 \to P_1)[2] \\ \qquad\quad \nearrow \\ \quad P_2[1] \\ \ \nearrow \\ P_2 \end{array} = \sigma_{Y_1}^{-1}\sigma_{Y_2}^{-1}(P_1)$$

$\Rightarrow$ This algorithm expresses each spherical as a distinguished braid word in generators given by stable sphericals.

The word depends on the stability condition!

Stable objects via braid words

Recall $\mathcal{C}_\Gamma$; associated to graph Γ



* The Grothendieck group $K_0(\mathcal{C}_\Gamma)$ is the root lattice of Γ.

* The pairing $\langle X, Y \rangle := \sum_i (-1)^i \dim \mathrm{Hom}^i(X, Y)$

descends to pairing on the root lattice.

Stable objects via braid words

The Grothendieck group $K_0(\mathcal{C}_\Gamma)$ is the root lattice of Γ & the pairing is induced by the hom (or Euler) pairing.

$\Rightarrow$ classes of spherical objects are real roots:

$$\langle s, s \rangle = \sum_i \dim \mathrm{Hom}^i(S,S) = 2.$$

Stable objects via braid words

The Grothendieck group $K_0(\mathcal{C}_\Gamma)$ is the root lattice of Γ & the pairing is induced by the hom (or Euler) pairing.

$\Rightarrow$ classes of spherical objects are real roots:

$$\langle S, S \rangle = \sum_i \dim \mathrm{Hom}^i(S,S) = 2.$$

Q: Is every real root representable by a spherical object?

Stable objects via braid words

Q: Is every real root representable by a spherical object?

A: Yes. If $\alpha = s_{i_1} s_{i_2} \cdots s_{i_k}(\alpha_m)$, then any lift $\sigma_{P_{i_1}}^{\pm} \sigma_{P_{i_2}}^{\pm} \cdots \sigma_{P_{i_k}}^{\pm} (P_m)$ has class α in $K_0(\mathcal{C}_r)$.

Stable objects via braid words

Q: Is every real root representable by a stable spherical object?

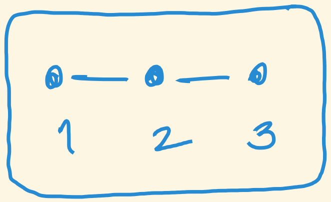
Stable objects via braid words

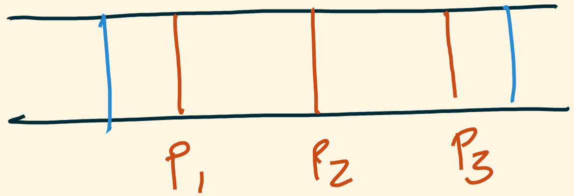
Q: Is every real root representable by a stable spherical object?

Thm [BDL] For a generic standard stability condition and a reduced expression $\alpha = s_{i_1} s_{i_2} \cdots s_{i_k}(\alpha_m)$, there is a unique stable lift among $\sigma_{i_1}^{\pm} \sigma_{i_2}^{\pm} \cdots \sigma_{i_k}^{\pm}(P_m)$.

Stable objects via braid words

Example : Type A_3





$$\alpha = S_3 S_2 (\alpha_1)$$

The only stable object of this class (up to shifts) is

$$\sigma_{P_3} \sigma_{P_2} (P_1) = P_3 \to P_2 \to P_1 .$$

Summary

Fix a stability condition on $\mathcal{C}_\Gamma$. We can:

1) express any spherical stable object as the braid group image of a simple spherical object

2) simplify any spherical object via twists in stable objects

3) measure asymptotic growth of spherical twists

Questions

1) Termination of the simplification algorithm for infinite type graphs?

2) Measure growth (categorical entropy) of arbitrary braids?

3) Other settings?

[See work of Auroux-Smith, Ishii-Uehara, Ishii-Ueda-Uehara, Smith-Wemyss, Hara-Wemyss, ...]

Thanks!