CATEGORICAL q-DEFORMED RATIONAL NUMBERS & COMPACTIFICATIONS OF STABILITY SPACE

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The big picture

$$B_\Gamma \circlearrowright V \quad \overset{\text{categorify}}{\rightsquigarrow} \quad B_\Gamma \circlearrowright \mathcal{C}$$

The big picture

(triangulated)

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The big picture

(triangulated)

$$B_\Gamma \circlearrowright V \quad \xrightarrow{\text{categorify}} \quad B_\Gamma \circlearrowright \mathcal{C}$$

$$\begin{array}{ccc} \mathrm{Stab}\,\mathcal{C} & \xrightarrow{\text{compactify}} & \overline{\mathrm{Stab}\,\mathcal{C}}^{\,?} \\ \circlearrowleft & & \circlearrowleft \\ B_\Gamma & & B_\Gamma \end{array}$$

The big picture

(triangulated)

$$B_\Gamma \circlearrowright V \xrightsquigarrow{\text{categorify}} B_\Gamma \circlearrowright \mathcal{C}$$

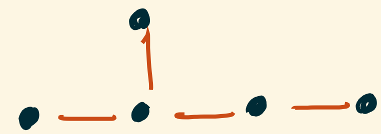
$$\begin{matrix} \mathrm{Stab}\,\mathcal{C} \\ \circlearrowleft \\ B_\Gamma \end{matrix} \xrightsquigarrow{\text{compactify}} \begin{matrix} \overline{\mathrm{Stab}}^{\,g}\,\mathcal{C} \\ \circlearrowleft \\ B_\Gamma \end{matrix}$$

Q: What is the topology of $\mathrm{Stab}\,\mathcal{C}$?

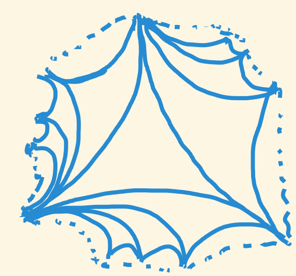
Q: What can we read off about Br_Γ from its action on $\overline{\mathrm{Stab}}^{\,g}\,\mathcal{C}$?

Plan

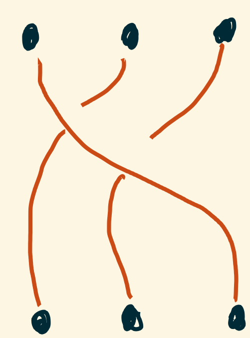
1. Generalities on $\mathcal{C}$, Stab, and the B_Γ-action



2. The family of compactifications



3. The three strand braid group



Categorical B_Γ action

$\mathscr{C}$ = 2-CY category of connected graph Γ

[categorifies Burau rep of B_Γ]

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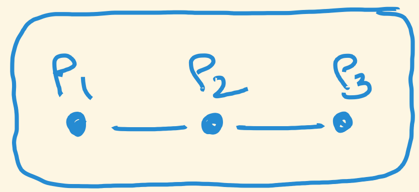
Important features:

- $\mathcal{C} = \langle P_i \mid i \text{ vertex} \rangle$

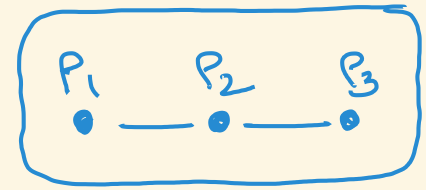


- Lots of spherical objects

$\Rightarrow$ lots of auto-equivalences.

Categorical B_Γ action

In particular, each P_i is spherical.



- $\sigma_{P_i} \circlearrowright \mathcal{C}$ is an autoequivalence;

- σ_{P_i} satisfy the braid relations. (of Γ)

$\Rightarrow B_\Gamma \circlearrowright \mathcal{C}$ (and yields Burau rep on Grothendieck group)

Bridgeland stability conditions & B_Γ-action

A stability condition τ is data on $\mathscr{C}$ that yields a family of metrics on $\mathscr{C}$: each arrow in $\mathscr{C}$ has a (τ, q)-length.

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This is called the "q-mass" of X wrt τ.

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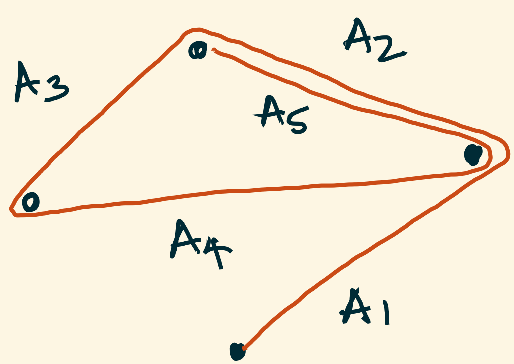
$X =$



, then

$$m_{q,\tau}(X) = \sum q^{\phi(A_i)} \cdot |A_i|$$

segments $\longleftrightarrow$ semistables

Bridgeland stability conditions & B_Γ-action

[Bridgeland] $\mathrm{Stab}\,\mathcal{C}$ is a complex manifold.

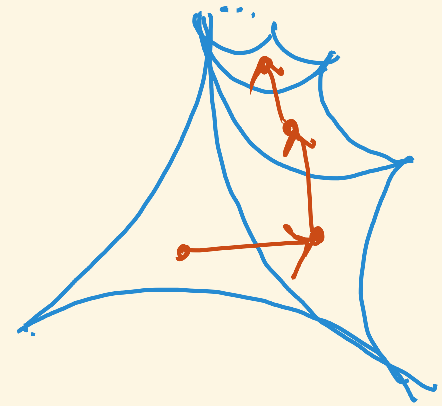
Since $B_\Gamma \circlearrowright \mathcal{C}$, we also have

$$B_\Gamma \circlearrowright \mathrm{Stab}\,\mathcal{C}.$$

Limiting operations on Stab $\mathcal{C}$

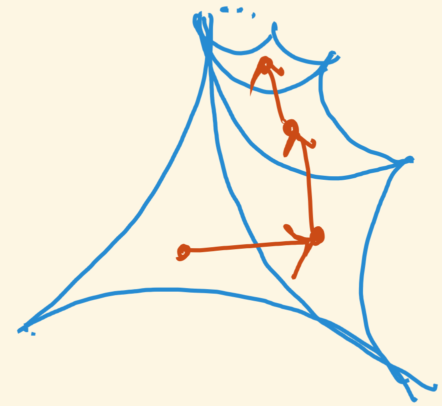
Limiting operations on Stab $\mathscr{C}$

① Fix $\beta \in B_r$ and $\tau \in \text{Stab}\,\mathscr{C}$.



Consider $\lim\limits_{n\to\infty} \beta^n \tau$.

Limiting operations on Stab $\mathscr{C}$



① Fix $\beta \in B_\Gamma$ and $\tau \in \operatorname{Stab}\mathscr{C}$.

Consider $\lim\limits_{n\to\infty} \beta^n \tau$.

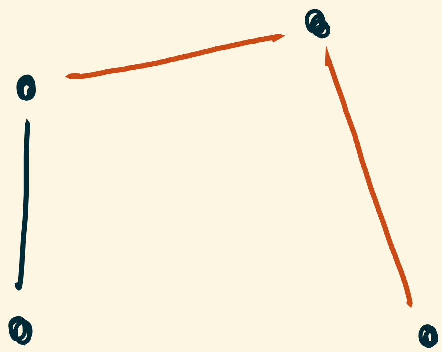
[BDL, BBL] Taking $\beta = \sigma_X$ for X spherical:

$$\lim_{n\to\infty} m_{\beta^n \tau, q}(Y) = q\text{-}\dim \operatorname{Hom}(X, Y)$$

upto simultaneous scalar

Limiting operations on Stab $\mathcal{C}$

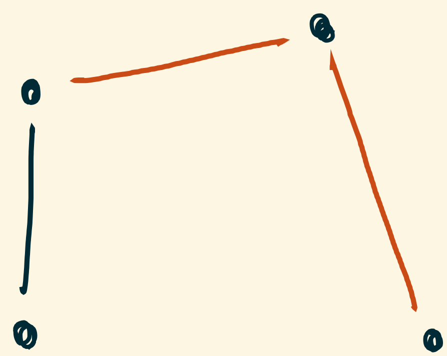
2



Shrink all but one of the simple semistables to zero.

Limiting operations on Stab $\mathcal{C}$

②



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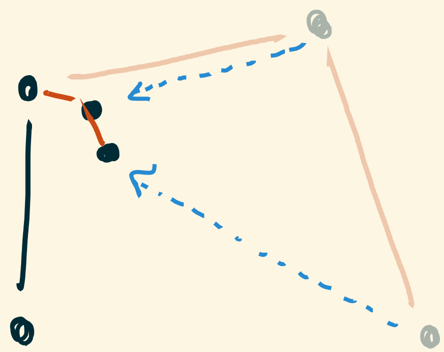

In the limit, the q-mass counts the "q-occurences" of the remaining semistable in any given object.

Limiting operations on $\mathrm{Stab}\,\mathcal{C}$

Moral: Limits may not make sense as stability conditions, but their q-masses make sense.

Limiting operations on Stab $\mathcal{C}$

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Mass map

$$\begin{aligned}
\operatorname{Stab}\mathcal{C} &\longrightarrow \mathbb{P}\mathbb{R}^{S} \\
\tau &\longmapsto [x \mapsto m_{q,\tau}(x)]/\sim
\end{aligned}$$

Mass map & compactification

$$\mathrm{Stab}\,\mathcal{C} \longrightarrow \mathbb{P}\mathbb{R}^{S}$$

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- [BDL, BBL] The mass map is injective, and $\overline{\mathrm{Stab}^{q}\,\mathcal{C}}$ is compact.

Mass map & compactification

- [BDL, BBL] The mass map is injective, and $\overline{\mathrm{Stab}}^q\mathcal{C}$ is compact.

- In the boundary, we see:

$\overline{\mathrm{hom}} := \lim_{n\to\infty} m_{\beta^n\tau, q}$ for β = spherical twist

occ := q-occurrences of a fixed semistable

General conjectures & questions

Q: $\overline{\mathrm{Stab}^q \mathcal{C}} \simeq$ closed ball ?

Q: $\overline{\text{hom}}$ & occ [+ linear combinations] recover a dense subset of the boundary sphere ?

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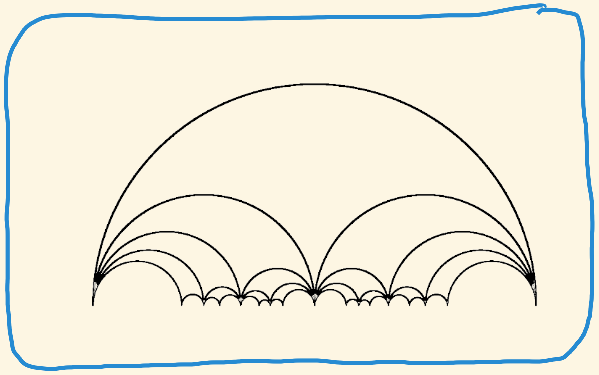
<u>Q</u>: $\overline{\mathrm{Stab}^q \mathcal{C}} \simeq$ closed ball ?

<u>Q</u>: $\overline{\mathrm{hom}}$ & occ [+ linear combinations] recover a dense subset of the boundary sphere ?

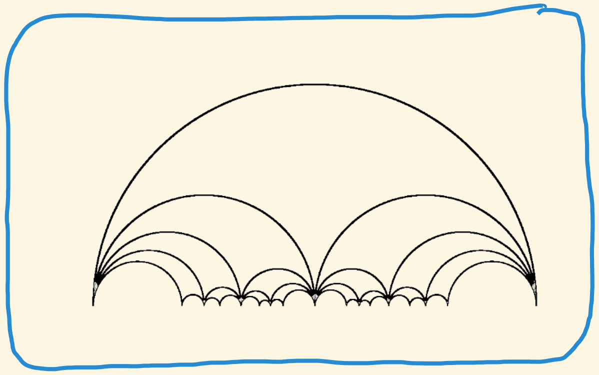
<u>Q</u>: What does this tell us about B_Γ ? What are the other points on the boundary ?

The story of the 3-strand braid group



The story of the 3-strand braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$



$$B_3 \longrightarrow PSL_2(\mathbb{Z})$$

$$\sigma_1 \longmapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \longmapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

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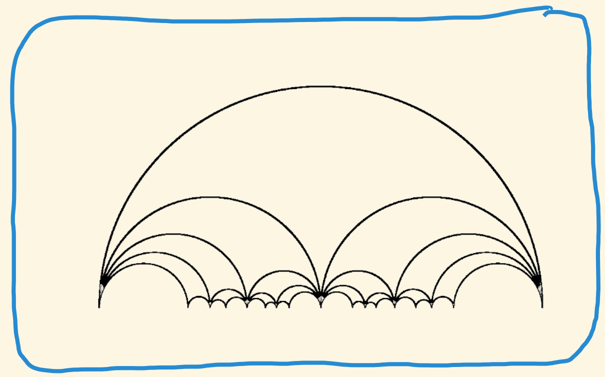

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- $PSL_2(\mathbb{Z})$, and hence B_3, acts on $\mathbb{C} \cup \{\infty\}$ by fractional linear transformations
- Action preserves $\mathbb{H}$ and $\mathbb{R} \cup \{\infty\}$

The story of the 3-strand braid group

For the remainder of the talk, take

$$\mathcal{C} = \mathcal{C}(\bullet\!-\!\!-\!\bullet) = \langle P_1, P_2 \rangle \circlearrowleft B_3$$

Fact:

$$\begin{array}{ccc} \mathrm{Stab}\,\mathcal{C} & \simeq & \mathbb{H} \\ \circlearrowright & & \circlearrowleft \\ B_3 & & B_3 \text{ via } PSL_2(\mathbb{Z}) \end{array}$$

The story of the 3-strand braid group

Take $\mathcal{C} = \mathcal{C}(\bullet - \bullet) = \langle P_1, P_2 \rangle \circlearrowleft B_3$

Thm [BDL]: For $q=1$:

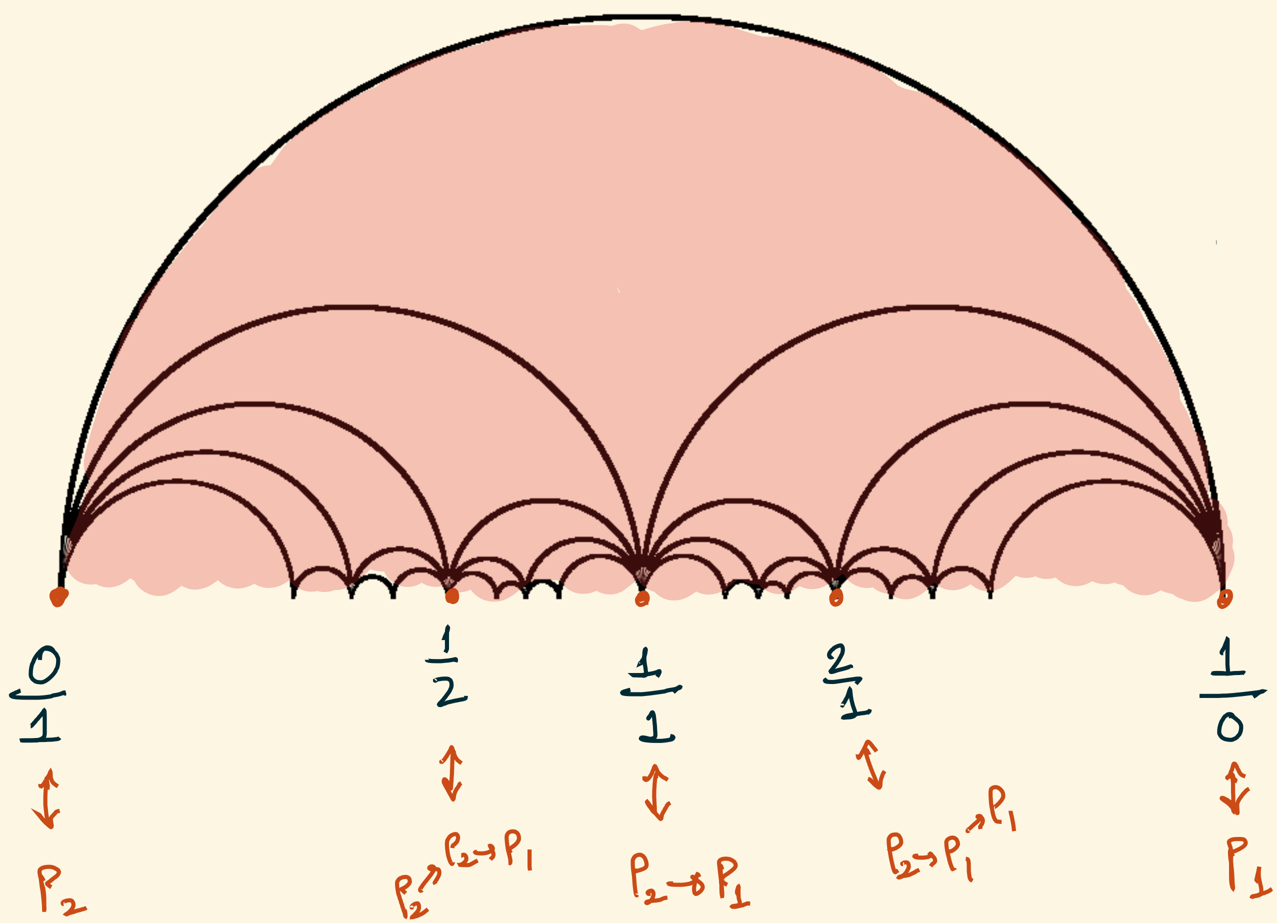
① $\overline{\hom}$ and occ coincide.

② $\overline{\hom}_X \mapsto \pm \dfrac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$ is a

B_3-equivariant bijection from the spherical objects of $\mathcal{C}$ to $\mathbb{Q} \cup \{\infty\}$

The lion functionals as rationals

Pictorially, at $q = 1$:



The q-deformed story for B_3

Thm [BBL] For an indeterminate q:

① $\overline{\hom}_X \mapsto \pm q^{()} \dfrac{\overline{\hom}_q(X, P_2)}{\overline{\hom}_q(X, P_1)}$ and

$\mathrm{occ}_X \mapsto \pm q^{()} \dfrac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$ are B_3-equivariant.

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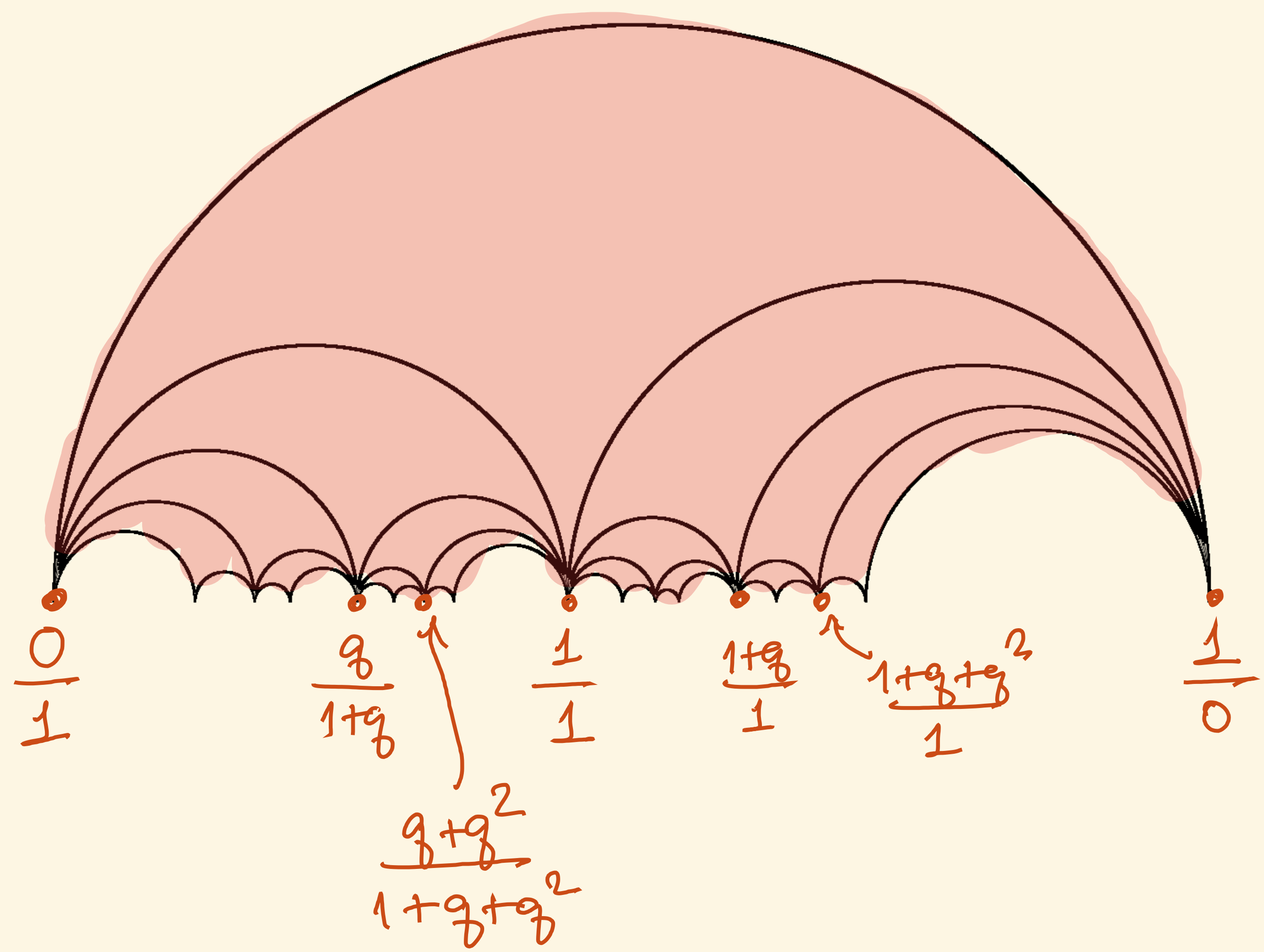
The B_3-action on the right is by fractional linear transformations via Burau matrices.

The q-deformed story for B_3

Pictorially, at $q \neq 1$:



The q-deformed story for $\mathbb{B}_3$

Thm [cont'd]

② $\pm q^{()} \dfrac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$ are exactly the q-deformed rationals of Morier-Genoud-Ovsienko.

③ $\pm q^{()} \dfrac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$ give a new q-deformation of $\mathbb{Q} \cup \{\infty\}$.

The q-deformed story for B_3

For $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$ corresponding to the spherical object X, set:

① $$\left[\frac{r}{s}\right]_q^{\#} := \pm q^{(\,)} \frac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$$ right q-deformed rational

② $$\left[\frac{r}{s}\right]_q^{\flat} := \pm q^{(\,)} \frac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$$ left q-deformed rational.

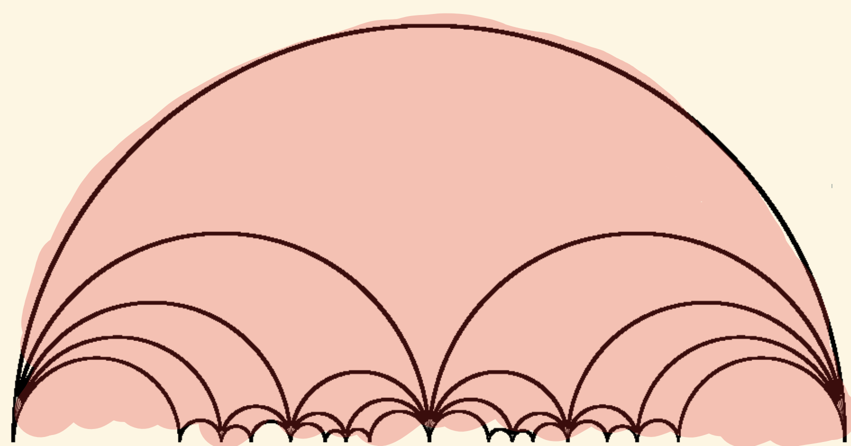
Specialising q

Now fix $0 < q < 1$.

Consider the ideal triangle with vertices $0, 1, \infty$.

[corresponds to a piece of stability space]

The $PSL_2(\mathbb{Z})$-orbit:



[$q=1$]

Specialising q

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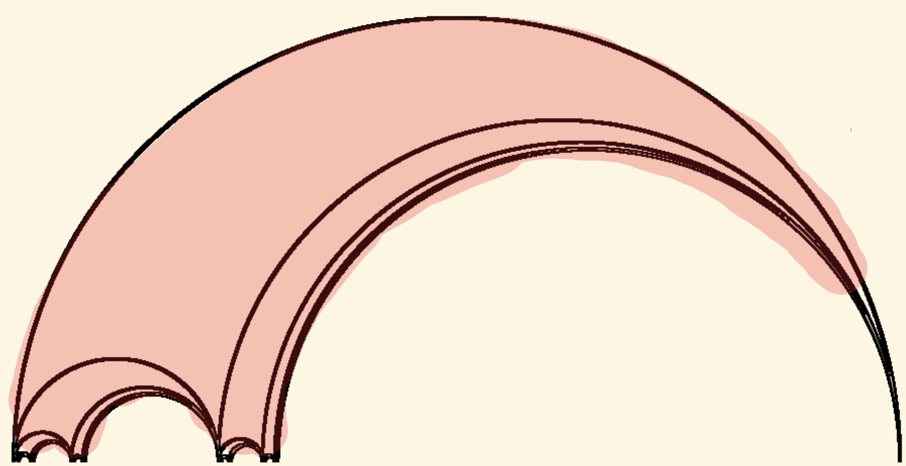
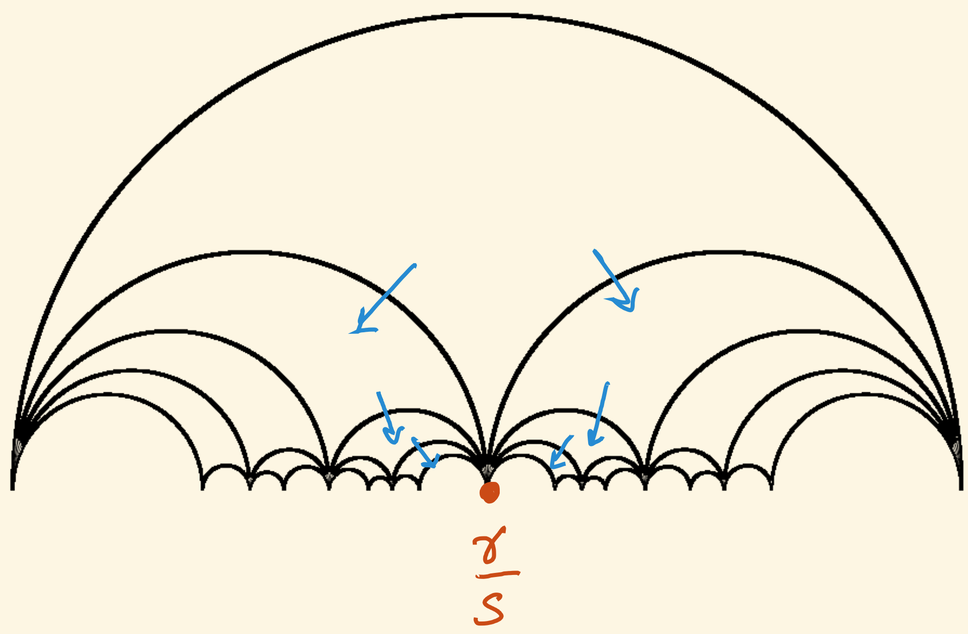
The $PSL_{2,q}(\mathbb{Z})$-orbit:



$[q = 0.3]$

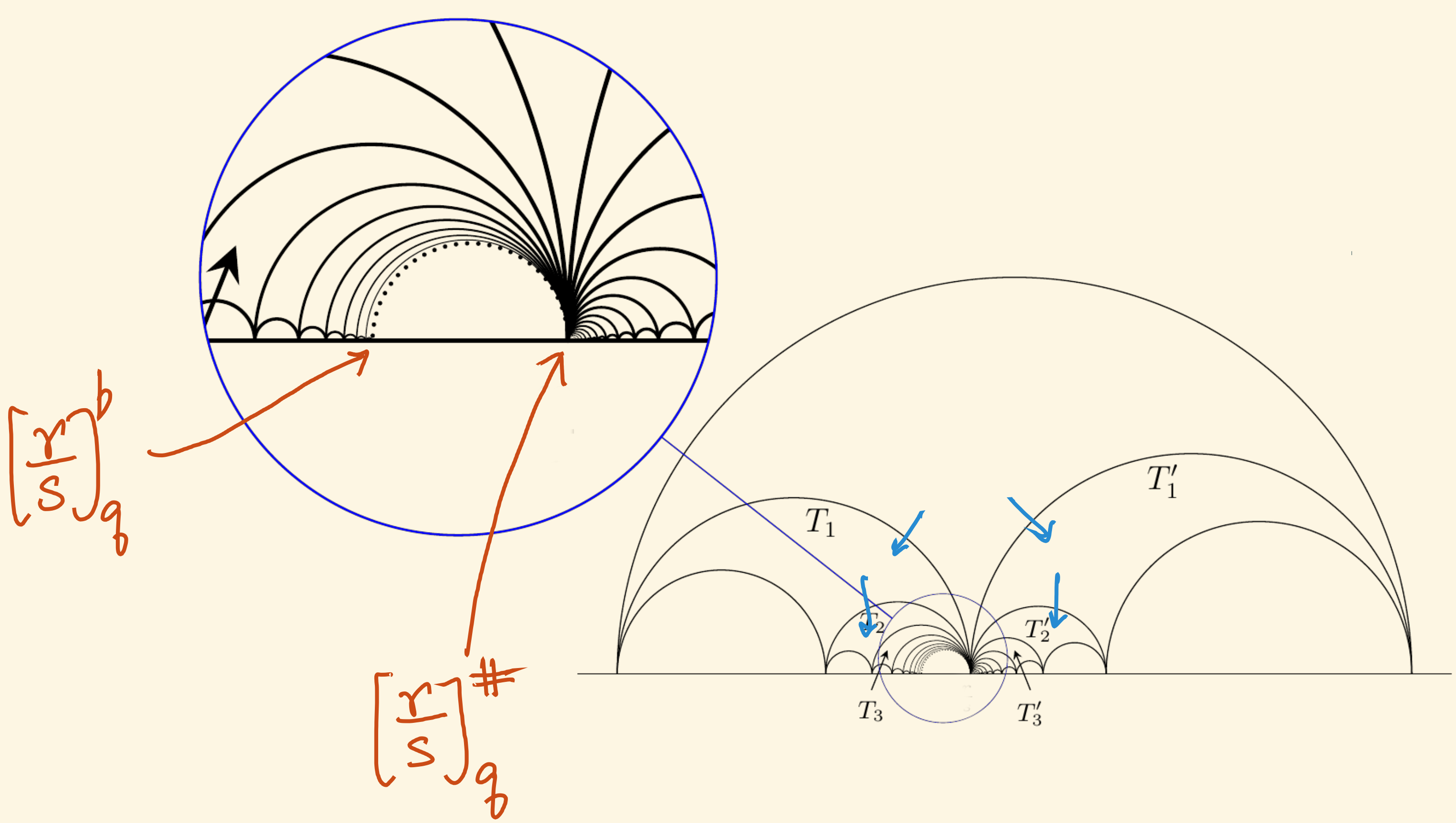
Specialising q

At $q=1$, left & right limits of Farey triangles agree.



Specialising q

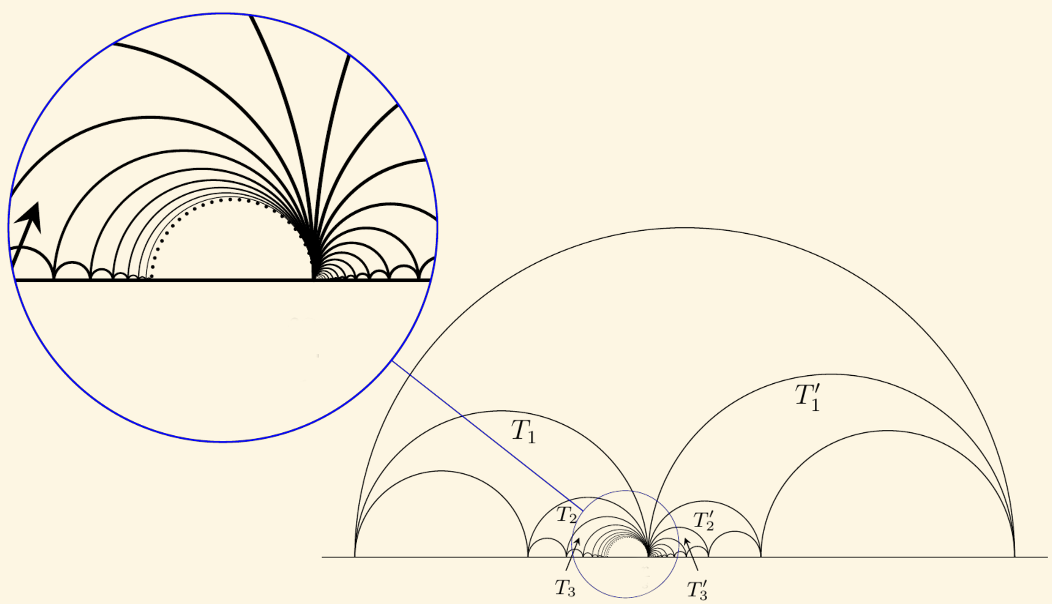
At $q \neq 1$, the left & right limits of Farey triangles <u>do not</u> agree – we get $\left[\frac{r}{s}\right]_q^\flat$ & $\left[\frac{r}{s}\right]_q^\sharp$!



Specialising q

At $q \neq 1$, the left & right limits of Farey triangles do not agree – we get $\left[\frac{r}{s}\right]^{\flat}_{q}$ & $\left[\frac{r}{s}\right]^{\sharp}_{q}$!

Moreover, the entire semicircle connecting them lies in the limit.



$\overline{\mathrm{Stab}}^q\,\mathcal{C}$ at a fixed positive q

Thm [B-Becker-Licata]

① The union of the closed semicircles $\left[\left[\frac{r}{s}\right]_q^\flat, \left[\frac{r}{s}\right]_q^\sharp\right]$ is dense in the boundary of $\overline{\mathrm{Stab}}^q\,\mathcal{C}$

② The remaining points of the boundary are exactly the "q-irrationals".

③ The boundary is homeomorphic to S^1.

Thank you!