Categorical $q_d$-deformed rational numbers & compactifications of stability space

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The big picture

$B_r C \xrightarrow{\text{categorify}} B_r C \&$
The big picture

$B_r \to V \xrightarrow{\text{categorify}} B_r \to C \to \mathcal{C}$ (triangulated)
The big picture

$B_r \xrightarrow{\text{V}} \xrightarrow{\text{categorify}} B_r \subset \mathcal{C}$

$\text{Stab } \mathcal{C} \xrightarrow{\text{compactify}} \frac{\text{Stab } \mathcal{C}}{\mathcal{C}} \xrightarrow{\gamma} \frac{\text{Stab } \mathcal{C}}{\mathcal{C}}$
The big picture

\[ B_r \rightarrow V \xrightarrow{\text{categorify}} B_r \rightarrow C \rightarrow \mathcal{C} \]

\[ \text{Stab} \mathcal{C} \xrightarrow{\text{compactify}} \overline{\text{Stab} \mathcal{C}} \]

Q: What is the topology of \( \text{Stab} \mathcal{C} \)?

Q: What can we read off about \( B_r \) from its action on \( \overline{\text{Stab} \mathcal{C}} \)?

(triangulated)
Plan

1. Generalities on $G$, Stab, and the $B_f$-action

2. The family of compactifications

3. The three-strand braid group
Categorical $B_\Gamma$ action

$\mathcal{C}$ = 2-CY category of connected graph $\Gamma$  
[categorifies Burau rep of $B_\Gamma$]
**Categorical $B\tau$ action**

$\mathcal{C} = 2$-cy category of connected graph $\Gamma$ 
[categorifies Burau rep of $B\tau$]

**Important features:**

- $\mathcal{C} = \{ P_i \mid i \text{ vertex} \}$

- Lots of spherical objects
  $\Rightarrow$ lots of auto-equivalences
Categorical $B_r$ action

In particular, each $P_i$ is spherical.

- $\sigma_{P_i} \in \mathcal{C}$ is an autoequivalence;
- $\sigma_{P_i}$ satisfy the braid relations (of $\Gamma$)

$\Rightarrow B_r \in \mathcal{C}$ (and yields Burau rep on Grothendieck group)
Bridgeland stability conditions & Br-action

A stability condition $\tau$ is data on $\mathcal{C}$ that yields a family of metrics on $\mathcal{C}$: each arrow in $\mathcal{C}$ has a $(\tau, q)$-length.
Bridgeland stability conditions & Br-action

A stability condition $Z$ is data on $C$ that yields a family of metrics on $C$: each arrow in $C$ has a $(Z,q)$-length.

The size of $X \in ob C$ is measured by “pulling tight to a geodesic” $0 \to X$. 

Bridgeland stability conditions & Br-action

The size of $X \in \mathcal{C}$ is measured by "pulling tight to a geodesic" $0 \to X$.

This is called the "$q$-mass" of $X$ wrt $\mathcal{C}$. 
Bridgeland stability conditions & $Br$-action

The size of $X \in \text{ob} \\mathcal{C}$ is measured by "pulling tight to a geodesic" $0 \rightarrow X$.

This is called the "$q$-mass" of $X$ wrt $\mathcal{C}$.

$$X = \quad \text{then} \quad M_{q, \mathcal{C}}(X) = \sum_\phi q^\phi \cdot |A_i|$$

segments $\leftrightarrow$ semistables
Bridgeland stability conditions & $B_r$-action

[Bridgeland] Stab $\mathcal{C}$ is a complex manifold.

Since $B_r \subset \mathcal{C}$, we also have $B_r \subset \text{Stab} \mathcal{C}$. 
Limiting operations on Stable 6
Limiting operations on $\text{Stab } \mathcal{C}$

1. Fix $\beta \in B_r$ and $\tau \in \text{Stab } \mathcal{C}$.
   Consider $\lim_{n \to \infty} \beta^n \tau$. 

\[\text{Diagram of a network with nodes and edges.}\]
Limiting operations on Stab $\mathcal{C}$

1. Fix $\beta \in B_\mathcal{F}$ and $\tau \in \text{Stab } \mathcal{C}$.

Consider $\lim_{n \to \infty} \beta^n \tau$.

[BDL, BBL] Taking $\beta = \xi^x$ for $X$ spherical:

$$\lim_{n \to \infty} \mathcal{M}_{\beta^n \tau, q} (\gamma) = q \cdot \text{dim } \text{Hom}(X, \gamma)$$

up to simultaneous scalar
Limiting operations on Stab C

Shrink all but one of the simple semistables to zero.
Limiting operations on Stab 6

Shrink all but one of the simple semistables to zero.

In the limit, the q-mass counts the “q-occurrences” of the remaining semistable in any given object.
Limiting operations on Stable C

Moral: Limits may not make sense as stability conditions, but their q-masses make sense.
Limiting operations on $\text{Stab}_e$

Moral: Limits may not make sense as stability conditions, but their q-masses make sense.

Mass map

$\text{Stab}_e \rightarrow \mathbb{P} \mathcal{R}_s^*$

$\mathbb{P} \mathcal{R}_s^* \rightarrow [x \mapsto m_{\text{q,m}}(x)]/\sim$
Mass map & compactification

\[ \text{Stab } \mathcal{C} \rightarrow \mathbb{P} \mathbb{R}^s \]

\[ \mathcal{Z} \rightarrow \left[ x \mapsto m_{q,z}(x) \right] /_{\sim} \]

- \text{[BDL, BBL]} The mass map is injective, and \( \text{Stab}^q \mathcal{C} \) is compact.
Mass map & compactification

- $[BDL, BBL]$ The mass map is injective, and $\overline{\text{Stable}}^q$ is compact.

- In the boundary, we see:

  $$\text{Hom} := \lim_{n \to \infty} W_{\beta, z, q}$$ for $\beta$ = spherical twist

  $$\text{occ} := q\text{-occurrences of a fixed semistable}$$
General conjectures & questions

Q: $\overline{\text{Stab}}^b \mathcal{C} = \text{closed ball}$?

Q: $\text{how \& occ [+] linear combinations}$
   recover a dense subset of the boundary sphere?
General conjectures & questions

Q: $\overline{\text{Stab}}^g \mathcal{C} \approx \text{closed ball}$?

Q: how & occ [$+ \text{linear combinations}$]
   recover a dense subset of the boundary sphere?

Q: What does this tell us about $B_{T}$?
   What are the other points on the boundary?
The story of the 3-strand braid group
The story of the 3-strand braid group

\[ B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle \]

\[ B_3 \twoheadrightarrow \text{PSL}_2(\mathbb{Z}) \]

\[ \sigma_1 \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \mapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \]
The story of the 3-strand braid group

\[ B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle \]

\[ B_3 \rightarrow \text{PSL}_2(\mathbb{Z}) \]

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• PSL$_2(\mathbb{Z})$, and hence $B_3$, acts on $\mathbb{C} \cup \partial \mathbb{C} \times \mathbb{S}^3$
  by fractional linear transformations

• Action preserves $\mathbb{H}$ and $\mathbb{R} \cup \partial \mathbb{C} \times \mathbb{S}^3$
The story of the 3-strand braid group

For the remainder of the talk, take

$$
\mathcal{C} = \mathcal{C}(\cdots) = \langle P_1, P_2 \rangle \circ B_3
$$

Fact:

$$
\text{Stab } \mathcal{C} \cong H
$$

$$
\begin{array}{cc}
\mathcal{C} & \circ \\
B_3 & B_3 \text{ via } \text{PSL}_2(\mathbb{Z})
\end{array}
$$
The story of the 3-strand braid group

Take $\mathcal{C} = \mathcal{C}(\ast \cdots \ast) = \langle P_1, P_2 \rangle \otimes B_3$

**Thm [BDL]:** For $q=\pm$:

1. $\overline{\text{hom}}$ and $\text{occ}$ coincide.

2. $\overline{\text{hom}}_X \mapsto \pm \overline{\text{hom}}(X, P_2) / \overline{\text{hom}}(X, P_1)$ is a $B_3$-equivariant bijection from the spherical objects of $\mathcal{C}$ to $\mathbb{Q} \langle \xi \delta \rangle$.
The \textbf{Liouville} functionals as rationals

Pictorially, at $q=1$: 
The $q$-deformed story for $B_3$

Thm [BBL] For an indeterminate $q$:

1. $\overline{\text{hom}}_q(x) \mapsto \pm q^\omega \overline{\text{hom}}_q(X, P_2)$ and

   $\text{occ}_q(x) \mapsto \pm q^\omega \frac{\text{occ}(P_2, X)}{\text{occ}(P_1, X)}$ are $B_3$-equivariant.
The $q$-deformed story for $B_3$

Thm [BBL] For an indeterminate $q$:

(1) \[ \overline{\text{hom}}_x \mapsto \pm q^{\alpha} \overline{\text{hom}}_q (X, P_2) \quad \text{and} \quad \overline{\text{hom}}_q (X, P_1) \]

\[ \text{occ}_x \mapsto \pm q^{\beta} \frac{\text{occ} (P_2, X)}{\text{occ} (P_1, X)} \]

are $B_3$-equivariant.

The $B_3$-action on the right is by fractional linear transformations via Burau matrices.
The $q$-deformed story for $B_3$

Pictorially, at $q \neq 1$: 

Diagram with mathematical expressions and arrows indicating relationships between terms.
The $q$-deformed story for $B_3$

Thm [cont'd]

2) $\pm q^{(1)} \frac{\text{occ}(p_2, x)}{\text{occ}(p_1, x)}$ are exactly the $q$-deformed rationals of Morier-Genoud - Ovsienko.

3) $\pm q^{(1)} \frac{\text{hom}(X, p_2)}{\text{hom}(X, p_1)}$ give a new $q$-deformation of $\mathbb{Q}U \tilde{\mathfrak{g}}^3$. 
The $q$-deformed story for $B_3$

For $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$ corresponding to the spherical object $X$, set:

1. $\left[ \frac{r}{s} \right]_q^\# := \pm q^{1} \frac{\text{occ}(P_2, X)}{\text{occ}(P_1, X)}$ \hspace{1cm} \text{right } q\text{-deformed rational}$

2. $\left[ \frac{r}{s} \right]_q^b := \pm q^{1} \frac{\hom(X, P_2)}{\hom(X, P_1)}$ \hspace{1cm} \text{left } q\text{-deformed rational}$
Specialising $q$

Now fix $0 < q < 1$.
Consider the ideal triangle with vertices $0, 1, \infty$ [corresponds to a piece of stability space].

The $\text{PSL}_2(\mathbb{Z})$-orbit:

$[q=1]$
Specialising \( q \)

Now fix \( 0 < q < 1 \).
Consider the ideal triangle with vertices \( 0, 1, \infty \).

The \( \text{PSL}_2, \mathbb{Q}_b(\mathbb{Z}) \) - orbit:

\[ q = 0.3 \]
Specialising $q$

At $q = 1$, left & right limits of Farey triangles agree.
Specialising \( q \)

At \( q \neq 1 \), the left & right limits of Farey triangles do not agree — we get \([\frac{r}{s}]^b_q \) & \([\frac{r}{s}]^*_q \)!
Specialising \( q \)

At \( q \neq 1 \), the left & right limits of Farey triangles do not agree - we get \( \left[ \frac{r}{s} \right]_q^b \) & \( \left[ \frac{r}{s} \right]_q^* \)!

Moreover, the entire semicircle connecting them lies in the limit.
\[
\text{Stab}^g S \text{ at a fixed positive } q
\]

Thm [B-Becker-Licata]

1. The union of the closed semicircles \([\frac{r}{s}]_q^b, [\frac{r}{s}]_q^a\) is dense in the boundary of \(\overline{\text{Stab}^g S}\).

2. The remaining points of the boundary are exactly the "q-irrationals".

3. The boundary is homeomorphic to \(S^1\).
Thank you!