CATEGORICAL q-DEFORMED RATIONAL NUMBERS & COMPACTIFICATIONS OF STABILITY SPACE

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\+ Louis Becker
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Anthony Licata

The big picture

$$B_\Gamma \circlearrowright V \quad \overset{\text{categorify}}{\rightsquigarrow} \quad B_\Gamma \circlearrowright \mathscr{C}$$

The big picture

(triangulated)

$$B_\Gamma \circlearrowright V_\Gamma \quad \overset{\text{categorify}}{\rightsquigarrow} \quad B_\Gamma \circlearrowright \mathcal{C}_\Gamma$$

The big picture

$$B_\Gamma \circlearrowright V_\Gamma \xrightarrow{\text{categorify}} B_\Gamma \circlearrowright \mathcal{C}_\Gamma$$

(triangulated)

$$\begin{matrix} \operatorname{Stab}\mathcal{C}_\Gamma \\ \circlearrowleft \\ B_\Gamma \end{matrix} \xrightarrow{\text{compactify}} \begin{matrix} \overline{\operatorname{Stab}\mathcal{C}_\Gamma}^{\,?} \\ \circlearrowleft \\ B_\Gamma \end{matrix}$$

The big picture

$$B_\Gamma \circlearrowright V_\Gamma \quad \overset{\text{categorify}}{\rightsquigarrow} \quad B_\Gamma \circlearrowright \mathcal{C}_\Gamma$$

(triangulated)

$$\begin{array}{c} \mathrm{Stab}\,\mathcal{C}_\Gamma \\ \circlearrowleft \\ B_\Gamma \end{array} \quad \overset{\text{compactify}}{\rightsquigarrow} \quad \begin{array}{c} \overline{\mathrm{Stab}}^{\,q}\,\mathcal{C}_\Gamma \\ \circlearrowleft \\ B_\Gamma \end{array}$$

Q: What is the topology of $\mathrm{Stab}\,\mathcal{C}_\Gamma$?

Q: What can we read off about Br_Γ from its action on $\overline{\mathrm{Stab}}^{\,q}\,\mathcal{C}_\Gamma$?

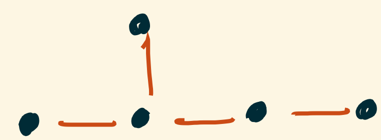
Tensor categories (aside)

Unfortunately, not really an ingredient in this talk!

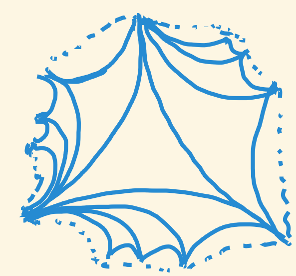
However, if $\mathcal{C}$ carries an action by a fusion category $\mathcal{F}$, it is fruitful to consider stability conditions respecting the action of $\mathcal{F}$ [e.g. work of E. Heng]

Plan

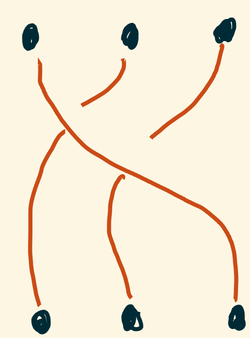
1. Generalities on $\mathcal{C}_\Gamma$, Stab, and the B_Γ-action



2. The family of compactifications



3. The three strand braid group



Categorical B_Γ action

$\mathcal{C}_\Gamma$ = 2-CY category of connected graph Γ

[categorifies Burau rep of B_Γ]

Categorical B_Γ action

$\mathcal{C}_\Gamma$ = 2-CY category of connected graph Γ
[categorifies Burau rep of B_Γ]

Constructed via zig-zag algebra of Γ
(a quotient of path algebra of Γ^{dbl}),
considered as dga.

$\mathcal{C}_\Gamma$ = homotopy category of projective modules:

$$K^b(A_\Gamma\text{-proj})$$

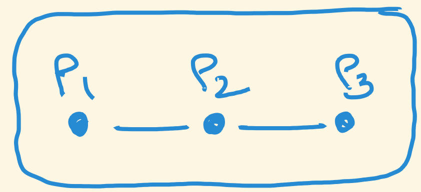
Categorical B_Γ action

$\mathcal{C}_\Gamma$ = 2-CY category of connected graph Γ

[categorifies Burau rep of B_Γ]

Important features:

- $\mathcal{C}_\Gamma = \langle P_i \mid i \text{ vertex} \rangle$

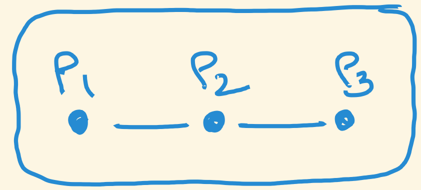


- Lots of spherical objects

$\Rightarrow$ lots of auto-equivalences.

Categorical B_Γ action

In particular, each P_i is spherical.



- $\sigma_{P_i} \circlearrowright \mathcal{C}_\Gamma$ is an autoequivalence;

- σ_{P_i} satisfy the braid relations (of Γ)

$\Rightarrow$ $B_\Gamma \circlearrowright \mathcal{C}_\Gamma$ (and yields Burau rep on Grothendieck group)

Bridgeland stability conditions & B_r-action

A stability condition τ is data on $\mathcal{C}_r$ that yields a family of metrics on $\mathcal{C}_r$;

each object X of $\mathcal{C}_r$ has a (q, τ)-mass.

Bridgeland stability conditions & B_r-action

A stability condition τ is data on $\mathcal{C}_r$ that yields a family of metrics on $\mathcal{C}_r$: each object X of $\mathcal{C}_r$ has a (q, τ)-mass.

Any τ consists of a bounded t-structure on $\mathcal{C}_r$ together with a stability function

$$Z : K(\mathcal{C}) \longrightarrow \mathbb{C}, \quad \text{additive on exact } \Delta\text{s},$$

such that $Z(X) \in \mathbb{H}$ if $X \in \heartsuit$,

satisfying the Harder–Narasimhan property.

Bridgeland stability conditions & B_Γ-action

As a result, τ yields, for any $X \in \mathrm{ob}\,\mathcal{C}_\Gamma$, a canonical Harder-Narasimhan filtration

$$
\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n = X \arrow[d] \\
& A_1 \arrow[ul, dotted] & A_2 \arrow[ul, dotted] & & A_n \arrow[ul, dotted]
\end{tikzcd}
$$

with τ-semistable pieces A_i.

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\end{tikzcd}$$

with τ-semistable pieces A_i.

Each semistable A has a modulus $|A|_\tau \in \mathbb{R}_{>0}$ and a phase $\phi_\tau(A) \in \mathbb{R}$.

Bridgeland stability conditions & B_Γ-action

The size of $X \in \mathrm{ob}\,\mathcal{C}_\Gamma$ is measured via the HN decomposition of X.

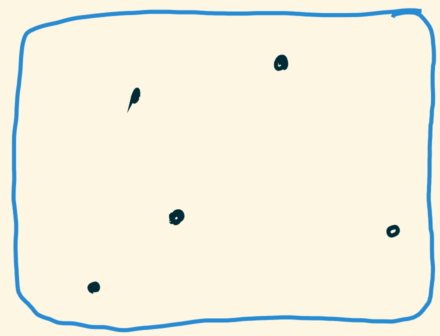
This is called the (q,τ)-mass of X.

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n = X \arrow[d] \\
A_1 \arrow[u, dotted] & A_2 \arrow[u, dotted] & & A_n \arrow[u, dotted] &
\end{tikzcd}
\implies
\boxed{m_{q,\tau}(X) := \sum q^{\phi_\tau(A_i)} \cdot |A_i|_\tau}$$

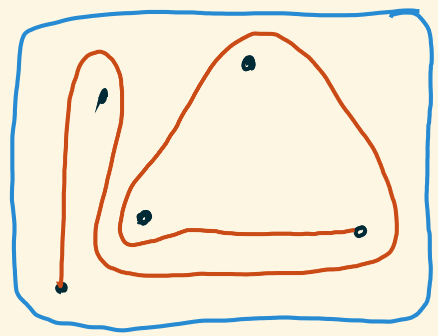
Bridgeland stability conditions & B_Γ-action

Illustration (accurate in type A!)

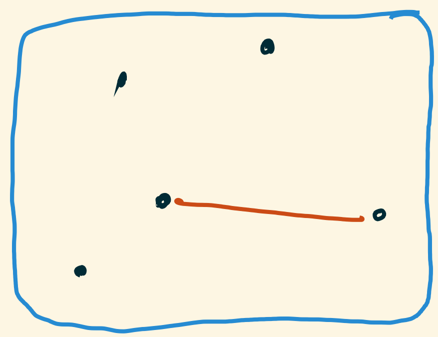
$\tau \longleftrightarrow$ configuration



$X \longleftrightarrow$



semistables $\longleftrightarrow$ segments



$|A|_\tau \longleftrightarrow$ length of segment

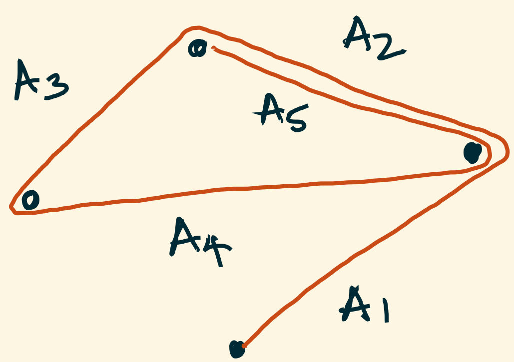
$\phi_\tau(A) \longleftrightarrow$ angle of segment to x-axis

Bridgeland stability conditions & B_r-action

The size of $X \in \mathrm{ob}\,\mathcal{C}_r$ is measured via the HN decomposition of X.

This is called the (q, τ)-mass of X.

$X =$



$\Rightarrow$

$$\boxed{m_{q,\tau}(X) := \sum q^{\phi_\tau(A_i)} \cdot |A_i|_\tau}$$

Bridgeland stability conditions & $\mathbb{B}_\Gamma$-action

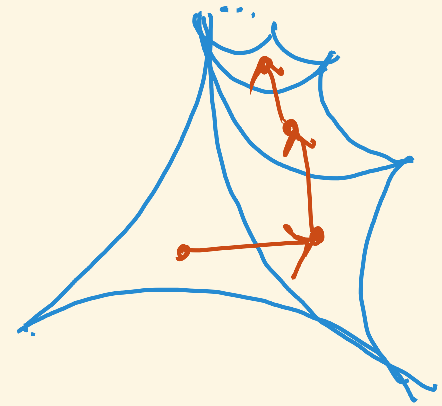
[Bridgeland] $\mathrm{Stab}\,\mathcal{C}_\Gamma$ is a complex manifold.

Since $\mathbb{B}_\Gamma \circlearrowright \mathcal{C}_\Gamma$, we also have

$$\mathbb{B}_\Gamma \circlearrowright \mathrm{Stab}\,\mathcal{C}_\Gamma .$$

Limiting operations on Stab $\mathcal{C}_r$

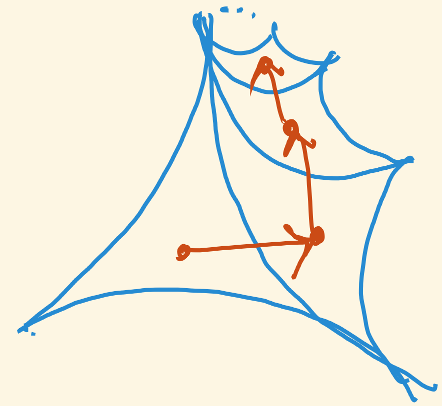
Limiting operations on $\operatorname{Stab} \mathcal{C}_r$



① Fix $\beta \in B_r$ and $\tau \in \operatorname{Stab} \mathcal{C}_r$.

Consider $\lim\limits_{n \to \infty} \beta^n \tau$.

Limiting operations on Stab $\mathcal{C}_r$



① Fix $\beta \in B_r$ and $\tau \in \operatorname{Stab} \mathcal{C}_r$.

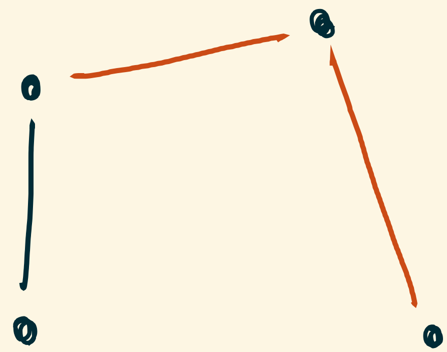
Consider $\lim_{n\to\infty} \beta^n \tau$.

[BDL, BBL] Taking $\beta = \sigma_X$ for X spherical :

$$\lim_{n\to\infty} m_{\beta^n\tau, q}(Y) = q\text{-}\dim \operatorname{Hom}(X, Y) \quad (*)$$

upto simultaneous scalar

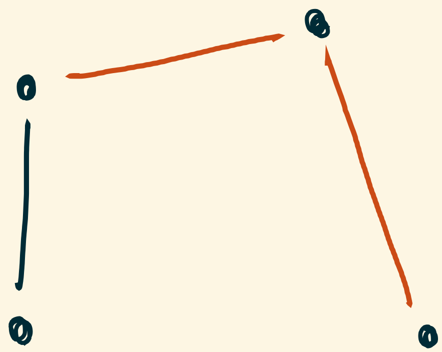
Limiting operations on Stab $\mathcal{C}_\Gamma$

② 

Shrink all but one of the simple semistables to zero.

Limiting operations on Stab $\mathcal{C}_\Gamma$

(2)



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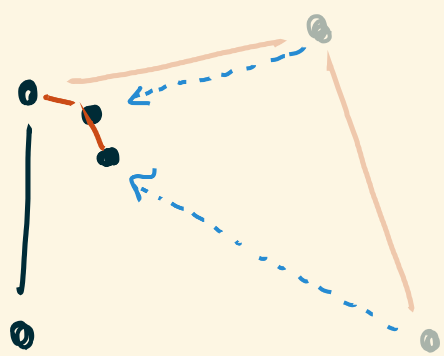

In the limit, the q-mass counts the "q-occurrences" of the remaining semistable in any given object.

Limiting operations on Stab $\mathcal{C}_r$

Moral: Limits may not make sense as stability conditions, but their q-masses make sense.

Limiting operations on $\mathrm{Stab}\,\mathcal{C}_r$

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Mass map

$$\mathrm{Stab}\,\mathcal{C}_r \longrightarrow \mathbb{P}\mathbb{R}^S$$

$$\tau \longmapsto [x \mapsto m_{q,\tau}(x)]/\sim$$

Mass map & compactification

$$\begin{aligned} \mathrm{Stab}\,\mathcal{C}_r &\longrightarrow \mathbb{P}\mathbb{R}^S \\ \tau &\longmapsto [x \mapsto m_{q,\tau}(x)]/\sim \end{aligned}$$

- [BDL, BBL] The mass map is injective, and $\overline{\mathrm{Stab}^q\,\mathcal{C}_r}$ is compact.

Mass map & compactification

- [BDL, BBL] The mass map is injective, and $\overline{\mathrm{Stab}^{q}\mathcal{C}_{\Gamma}}$ is compact.

- In the boundary, we see:

$\overline{\mathrm{hom}} := \lim_{n\to\infty} m_{\beta^n \tau, q}$ for β = spherical twist

occ := q-occurrences of a fixed semistable

General conjectures & questions

Q: $\overline{\mathrm{Stab}}^q\, \mathcal{C}_\Gamma \simeq$ closed ball ?

Q: $\overline{\mathrm{hom}}$ & occ [+ linear combinations] recover a dense subset of the boundary sphere ?

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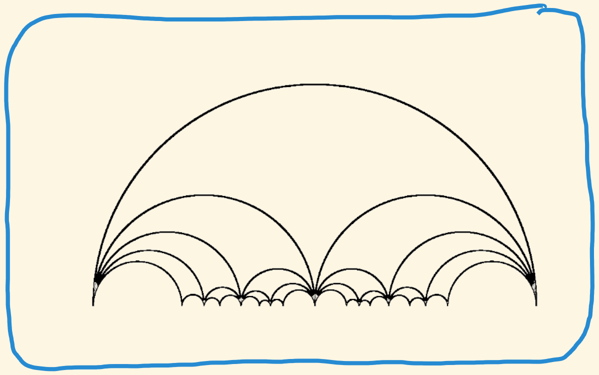
Q: $\overline{\mathrm{Stab}^q \mathcal{C}_\Gamma} \simeq$ closed ball ?

Q: $\overline{\text{hom}}$ & occ [+ linear combinations] recover a dense subset of the boundary sphere ?

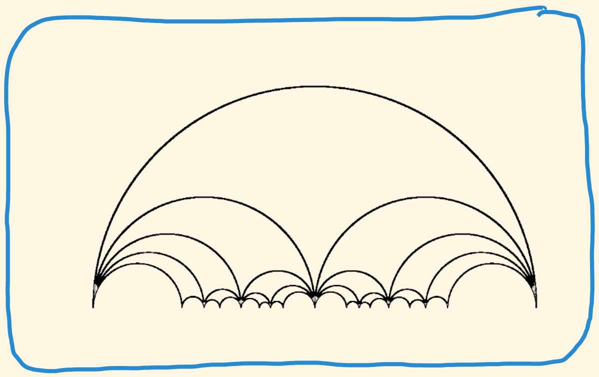
Q: What does this tell us about B_Γ ? What are the other points on the boundary ?

The story of the 3-strand braid group



The story of the 3-strand braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$



$$B_3 \longrightarrow PSL_2(\mathbb{Z})$$

$$\sigma_1 \longmapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \longmapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

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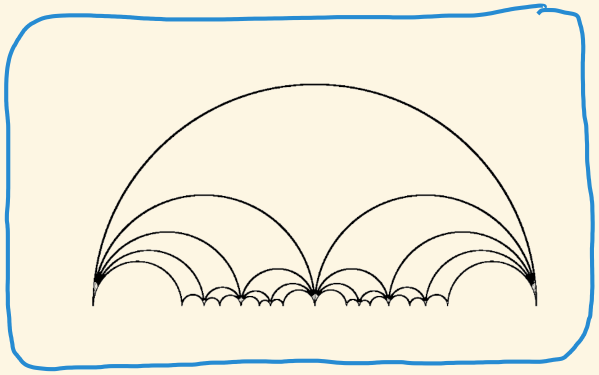

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- $PSL_2(\mathbb{Z})$, and hence B_3, acts on $\mathbb{C} \cup \{\infty\}$ by fractional linear transformations
- Action preserves $\mathbb{H}$ and $\mathbb{R} \cup \{\infty\}$

The story of the 3-strand braid group

For the remainder of the talk, take

$$\mathcal{C} = \mathcal{C}(\bullet - \bullet) = \langle P_1, P_2 \rangle \circlearrowleft B_3$$

Fact:

$$\begin{array}{ccc} \operatorname{Stab} \mathcal{C} & \simeq & \mathbb{H} \\ \circlearrowright & & \circlearrowleft \\ B_3 & & B_3 \text{ via } PSL_2(\mathbb{Z}) \end{array}$$

The story of the 3-strand braid group

Take $\mathcal{C} = \mathcal{C}(\bullet - \bullet) = \langle P_1, P_2 \rangle \circlearrowright B_3$

Thm [BDL]: For $q=1$:

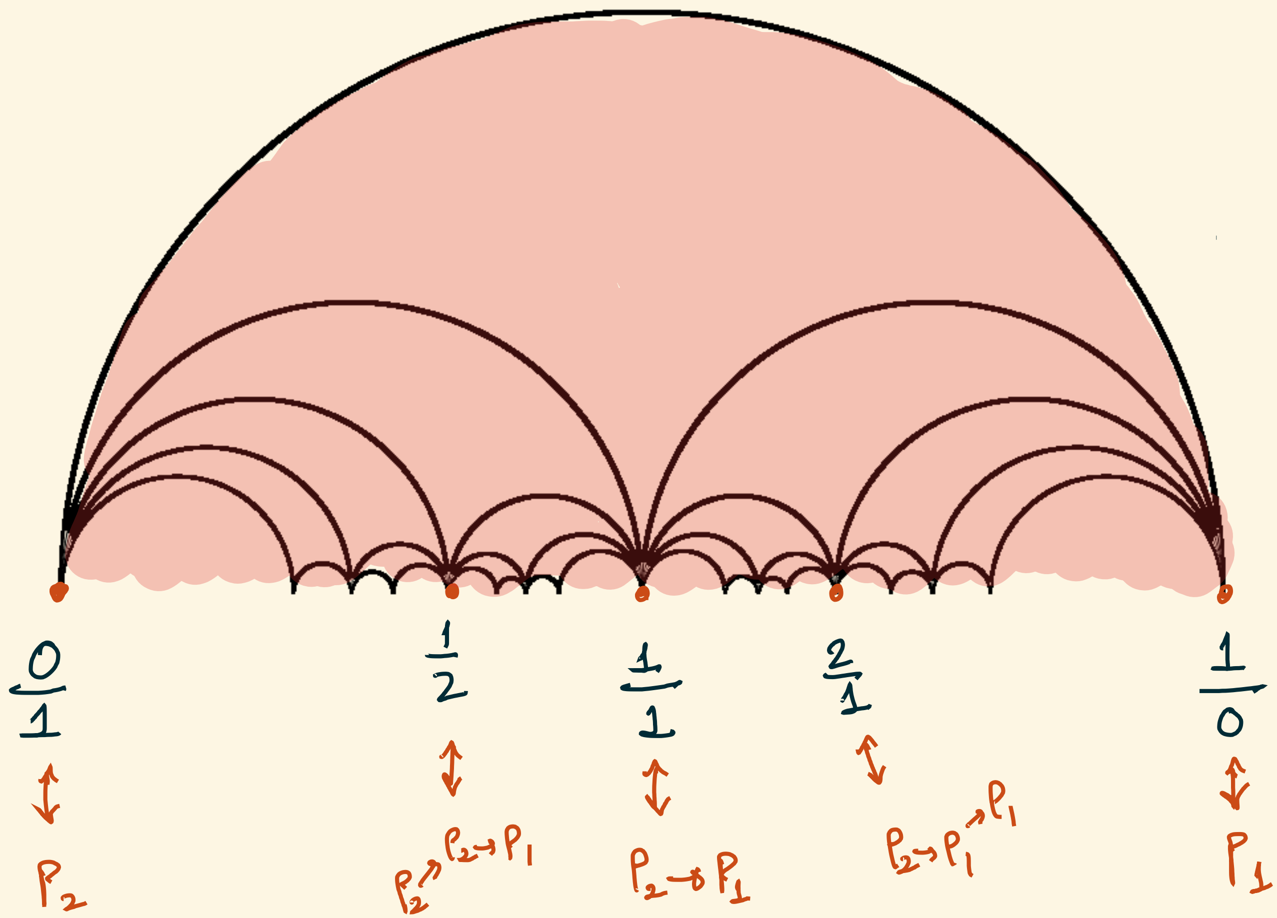
① $\overline{\hom}$ and occ coincide.

② $\overline{\hom}_X \mapsto \pm \dfrac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$ is a

B_3-equivariant bijection from the spherical objects of $\mathcal{C}$ to $\mathbb{Q} \cup \{\infty\}$

The lion functionals as rationals

Pictorially, at $q = 1$:



The q-deformed story for B_3

Thm [BBL] For an indeterminate q:

① $$\overline{\hom}_X \mapsto \pm q^{(\,)} \frac{\overline{\hom}_q(X, P_2)}{\overline{\hom}_q(X, P_1)}$$ and

$$\mathrm{occ}_X \mapsto \pm q^{(\,)} \frac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$$ are B_3-equivariant.

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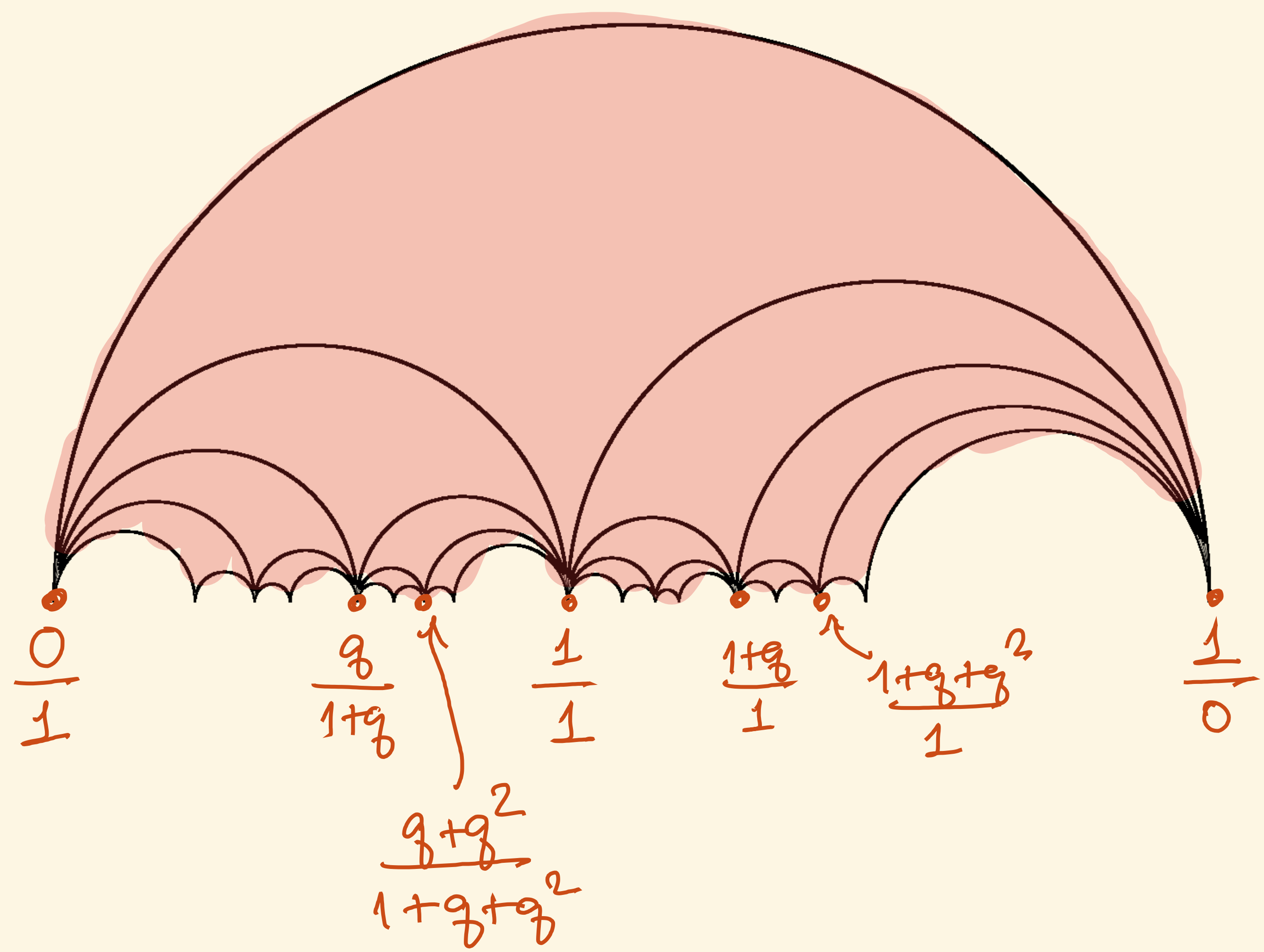
$$\mathrm{occ}_X \mapsto \pm q^{(\,)} \frac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$$ are B_3-equivariant.

The B_3-action on the right is by fractional linear transformations via Burau matrices.

The q-deformed story for B_3

Pictorially, at $q \neq 1$:



The q-deformed story for B_3

Thm [cont'd]

② $\pm q^{()} \dfrac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$ are exactly the q-deformed rationals of Morier-Genoud-Ovsienko.

③ $\pm q^{()} \dfrac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$ give a new q-deformation of $\mathbb{Q} \cup \{\infty\}$.

The q-deformed story for B_3

For $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$ corresponding to the spherical object X, set:

① $$\left[\frac{r}{s}\right]^{\#}_{q} := \pm q^{(\,)} \frac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$$ right q-deformed rational

② $$\left[\frac{r}{s}\right]^{b}_{q} := \pm q^{(\,)} \frac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$$ left q-deformed rational.

A word about q-rationals

The classical (right) q-rationals are defined by deforming continued-fraction expansion. They have a variety of fascinating properties.

Our homological interpretation naturally produces the left q-rationals; these should also satisfy similar properties.

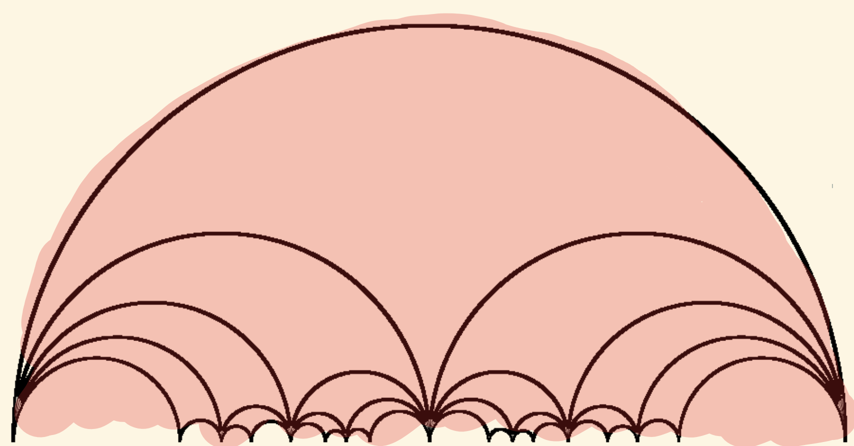
Specialising q

Let $q = 1$.

Consider the ideal triangle with vertices $0, 1, \infty$.

[corresponds to a piece of stability space]

The $PSL_2(\mathbb{Z})$-orbit:



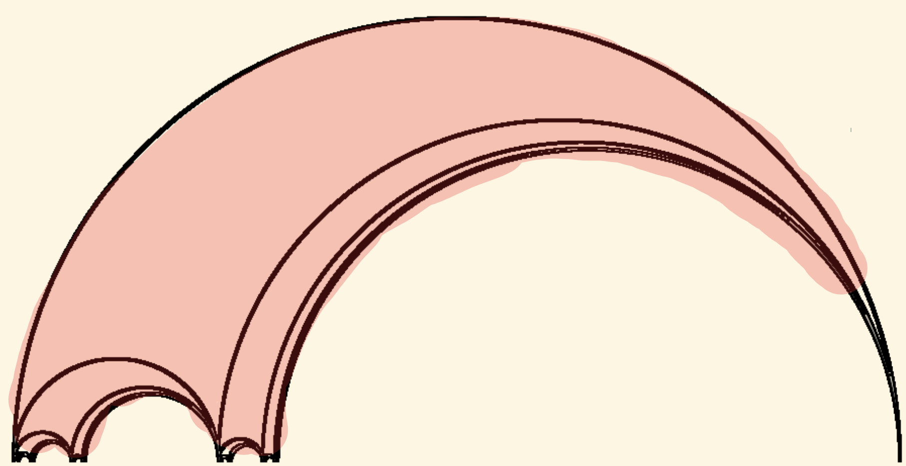
[$q=1$]

Specialising q

Now fix $0 < q < 1$.

Consider the ideal triangle with vertices $0, 1, \infty$.

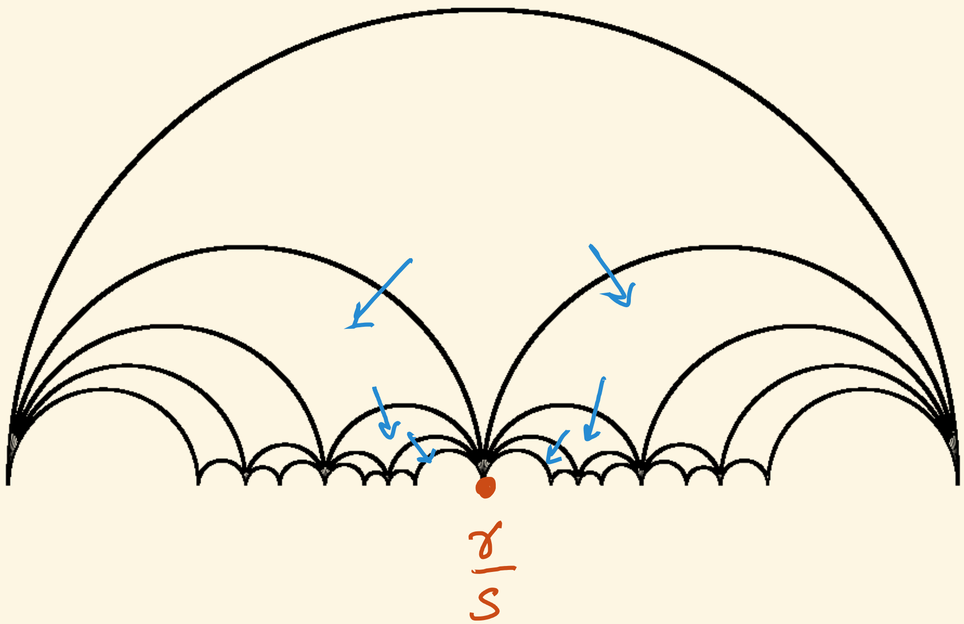
The $PSL_{2,q}(\mathbb{Z})$-orbit:



$[q = 0.3]$

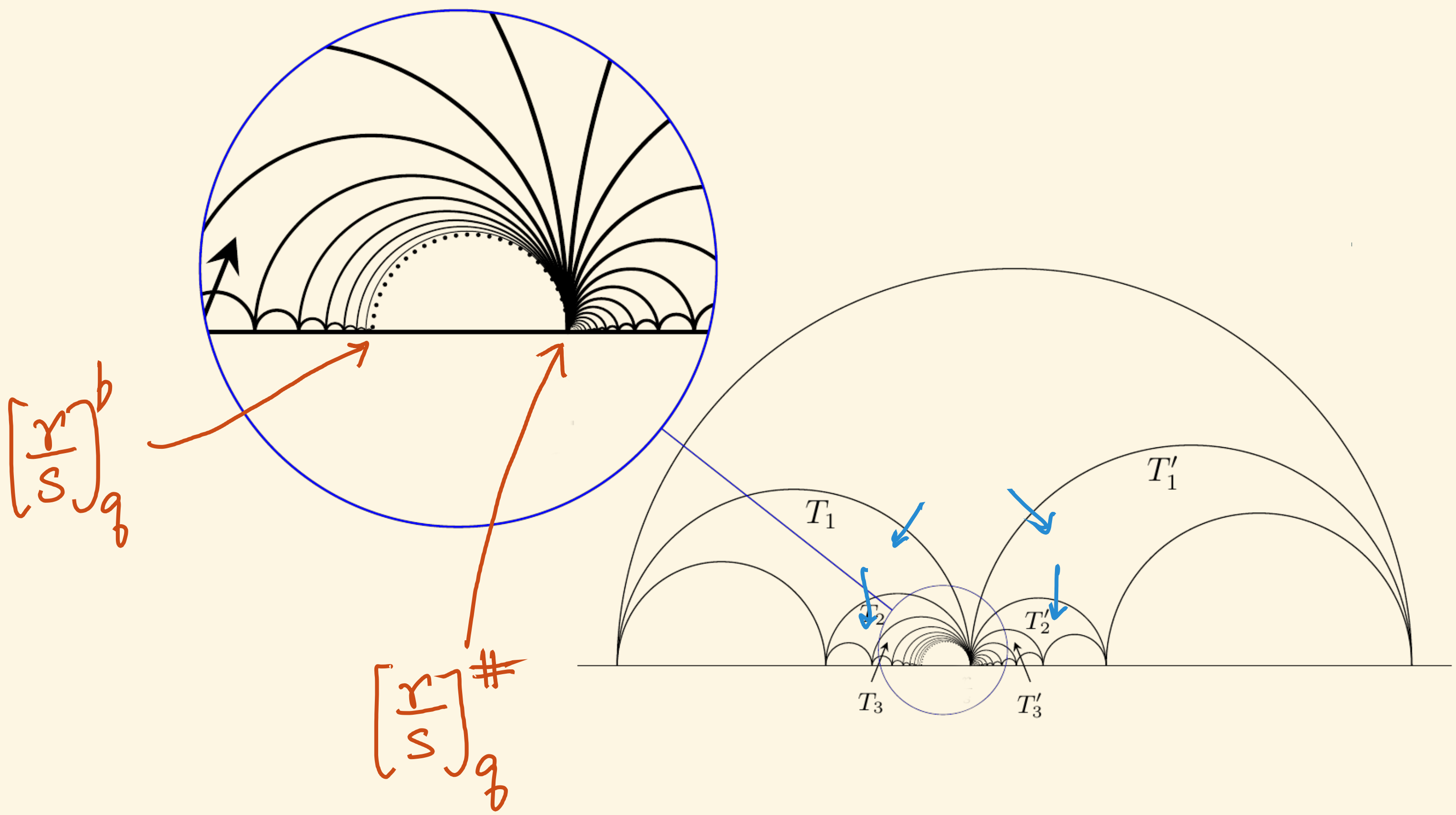
Specialising q

At $q = 1$, left & right limits of Farey triangles agree.



Specialising q

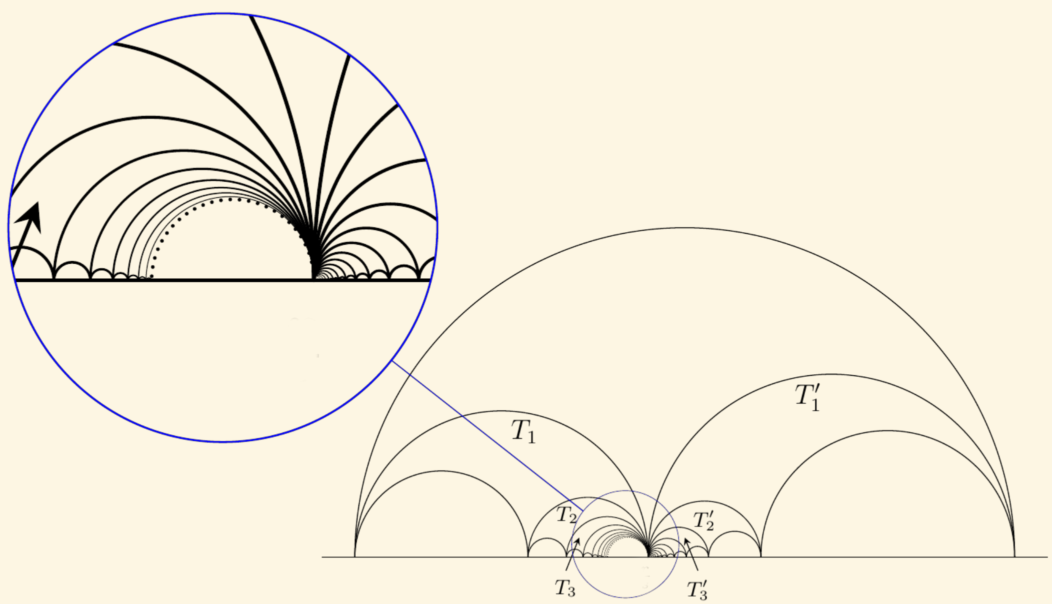
At $q \neq 1$, the left & right limits of Farey triangles do not agree – we get $\left[\frac{r}{s}\right]_q^{\flat}$ & $\left[\frac{r}{s}\right]_q^{\sharp}$!



Specialising q

At $q \neq 1$, the left & right limits of Farey triangles do not agree – we get $\left[\frac{r}{s}\right]^\flat_q$ & $\left[\frac{r}{s}\right]^\sharp_q$!

Moreover, the entire semicircle connecting them lies in the limit.



$\overline{\mathrm{Stab}}^q \mathcal{C}$ at a fixed positive q

Thm [B-Becker-Licata]

① The union of the closed semicircles $\left[\left[\frac{r}{s}\right]_q^{\flat}, \left[\frac{r}{s}\right]_q^{\sharp} \right]$ is dense in the boundary of $\overline{\mathrm{Stab}}^q \mathcal{C}$

② The remaining points of the boundary are exactly the "q-irrationals".

③ The boundary is homeomorphic to S^1.

Thank you!