CATEGORICAL q-DEFORMED RATIONAL NUMBERS VIA BRIDGELAND STABILITY CONDITIONS

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\+ Louis Becker,

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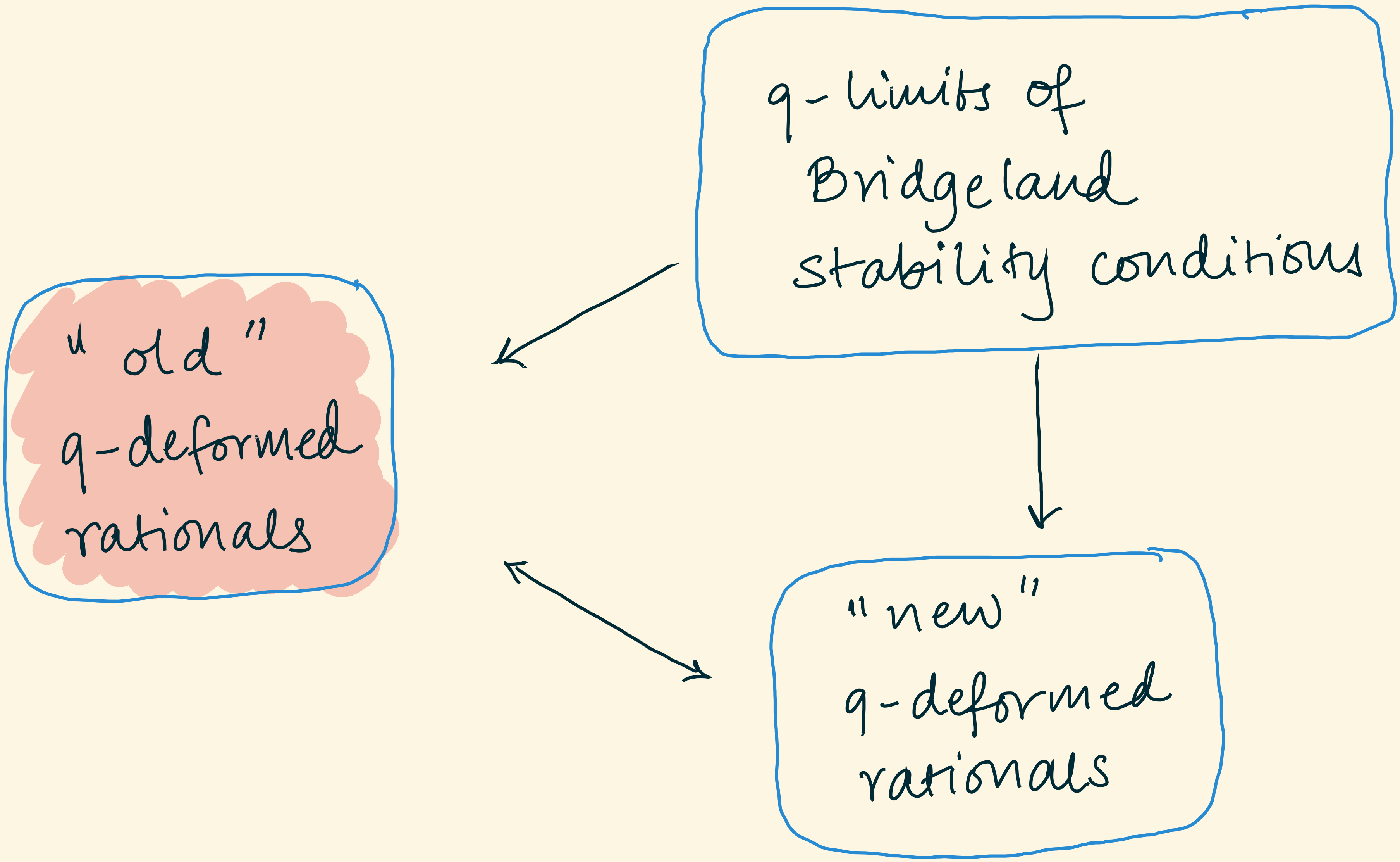
Outline

q-limits of Bridgeland stability conditions

"old" q-deformed rationals

"new" q-deformed rationals

Outline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Fractional linear action of B_3

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

There is a homomorphism

$$B_3 \to PSL_2(\mathbb{Z}) :$$

$$\sigma_1 \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \mapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Fractional linear action of B_3

$PSL_2(\mathbb{Z})$ acts on $\mathbb{R} \cup \{\infty\}$ via fractional linear transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \left(\frac{r}{s}\right) := \frac{ar+bs}{cr+ds}$$

$\Rightarrow$ * B_3 acts on $\mathbb{R} \cup \{\infty\}$.

* The action preserves $\mathbb{Q} \cup \{\infty\}$.

Fractional linear action of B_3

Can be realised via continued fractions.

Let $\dfrac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots a_{2n}}}$

Then

$$\boxed{\frac{r}{s} = \sigma_1^{-a_1} \sigma_2^{a_2} \sigma_1^{-a_3} \sigma_2^{a_4} \cdots \sigma_1^{-a_{2n-1}} \sigma_2^{a_{2n}} (\infty)}$$

Classical (right) q-deformed rationals

Consider deformed matrices:

$$\sigma_{1,q} := \begin{bmatrix} q^{-1} & -q^{-1} \\ 0 & 1 \end{bmatrix}, \quad \sigma_{2,q} := \begin{bmatrix} 1 & 0 \\ 1 & q^{-1} \end{bmatrix}$$

These generate a copy of B_3 in

$$PSL_2(\mathbb{Z}[q^{\pm}]).$$

Classical (right) q-deformed rationals

Let $\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots a_{2n}}}$

The corresponding right q-deformation is:

$$\boxed{\left[\frac{r}{s}\right]^{\#}_{q} = \sigma_{1,q}^{-a_1}\, \sigma_{2,q}^{a_2} \cdots \sigma_{1,q}^{-a_{2n-1}}\, \sigma_{2,q}^{a_{2n}} (\infty).}$$

[Morier-Genoud – Ovsienko]

Left q-deformed rationals

Let $\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots a_{2n}}}$

The corresponding left q-deformation is:

$$\boxed{\left[\frac{r}{s}\right]^b_q = \sigma_{1,q}^{-a_1} \sigma_{2,q}^{a_2} \cdots \sigma_{1,q}^{-a_{2n-1}} \sigma_{2,q}^{a_{2n}} \left(\frac{1}{1-q}\right).}$$

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Categorical interlude

$\mathscr{C}$ = 2-CY category of connected graph Γ

[categorifies Burau rep of B_Γ]

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$\mathcal{C}$ = 2-CY category of connected graph Γ

[categorifies Burau rep of B_Γ]

[constructed, e.g. via Ginzburg dga of doubled quiver of Γ, or dg-modules over zig-zag algebra of Γ. More generally, one may consider N-CY & $\mathbb{X}$-CY versions, but we only take N=2.]

Categorical interlude

$\mathcal{C}$ = 2-CY category of connected graph Γ

[categorifies Burau rep of B_r]

Main example for this talk:

$\Gamma = \bullet\!\!-\!\!\!-\!\!\bullet$ (A_2 graph)

& $B_r = B_3$

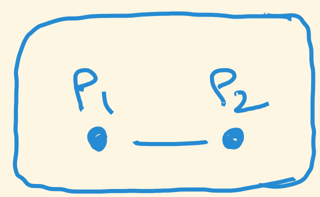
Categorical interlude

$\mathcal{C}$ = 2-CY category of connected graph Γ

[categorifies Burau rep of B_Γ]

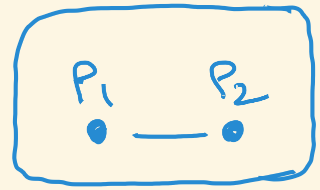
Important features:

- $\mathcal{C} = \langle P_i \mid i \text{ vertex} \rangle$



- More precisely, $\mathcal{C}$ = full triangulated subcategory (closed under iso.) of $K(A\text{-dgm})$ generated by $\langle P_i \rangle$, where A = zig-zag algebra.

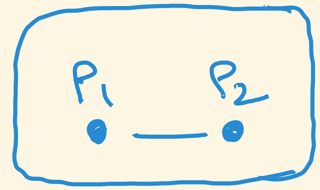
Categorical B_r action



The objects P_1 & P_2 are "spherical":

$$\mathrm{End}^m(P_i) = \begin{cases} \mathbb{C} & \text{if } m = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

Categorical B_Γ action



The objects P_1 & P_2 are "spherical":

$$\mathrm{End}^m(P_i) = \begin{cases} \mathbb{C} & \text{if } m = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

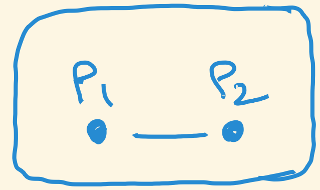
Any spherical object X defines autoequivalence $\sigma_X : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$.

In particular, $\sigma_{P_1}, \sigma_{P_2} : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$.

[Seidel-Thomas]

Categorical B_Γ action

The functors σ_{P_1} & σ_{P_2} braid:



$$\sigma_{P_1}\sigma_{P_2}\sigma_{P_1} \simeq \sigma_{P_2}\sigma_{P_1}\sigma_{P_2}$$

Therefore we have a (weak) action of B_3 on $\mathcal{C}_{A_2}$ with

$$\sigma_1 \mapsto \sigma_{P_1} \quad \& \quad \sigma_2 \mapsto \sigma_{P_2}.$$

Bridgeland stability conditions & B_Γ-action

We will encounter q-rationals again by taking "limiting q-sizes" of objects in $\mathcal{C}$.

These are provided by Bridgeland stability conditions.

Bridgeland stability conditions & B_Γ-action

A stability condition τ is data on $\mathcal{C}$ that yields a family of metrics on $\mathcal{C}$: each arrow in $\mathcal{C}$ has a length.

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The size of $X \in \mathrm{ob}\,\mathcal{C}$ is measured by "pulling tight" to a geodesic path

$$0 \to X_1 \to X_2 \to X_3 \to X$$

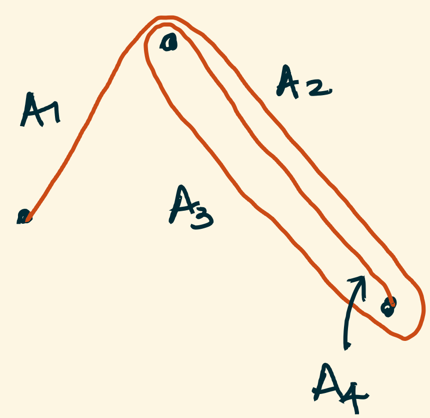
with "semistable" segments.

Bridgeland stability conditions & B_Γ-action

The size of $X \in \operatorname{ob} \mathcal{C}$ is measured by "pulling tight" to a geodesic $0 \to X$.

This is called the "q-mass" of X wrt τ.

If

$X =$ 

, then

$$m_{q,\tau}(X) = \sum q^{\phi(A_i)} \cdot |A_i|$$

Bridgeland stability conditions & B_Γ-action

[Bridgeland] $\mathrm{Stab}\,\mathcal{C}$ is a complex manifold.

Since $B_\Gamma \circlearrowright \mathcal{C}$, we also have

$$B_\Gamma \circlearrowright \mathrm{Stab}\,\mathcal{C}.$$

Bridgeland stability conditions & B_Γ-action

[Bridgeland] Stab $\mathscr{C}$ is a complex manifold.

When $\Gamma = A_2$, Stab $\mathscr{C}$ (modulo $\mathbb{C}$-action) is homeomorphic to the upper half plane.

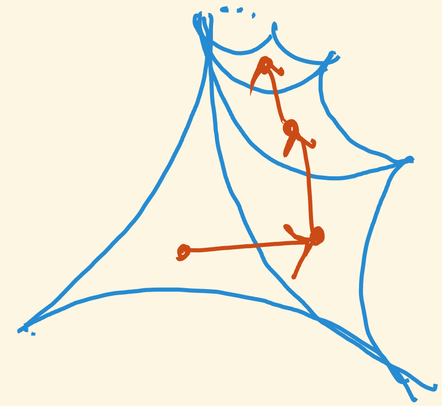
Under this correspondence, B_3 acts by fractional linear maps via $PSL_2(\mathbb{Z})$.

Limiting operations on Stab $\mathcal{C}$

Limiting operations on $\mathrm{Stab}\,\mathcal{C}$

① Fix $\beta \in B_\Gamma$ and $\tau \in \mathrm{Stab}\,\mathcal{C}$.

Consider $\lim\limits_{n \to \infty} \beta^n \tau$.



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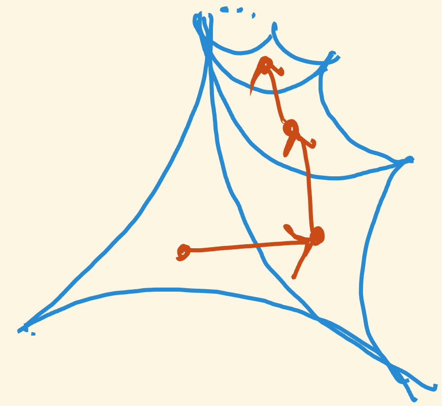


Consider $\lim\limits_{n\to\infty} \beta^n \tau$.

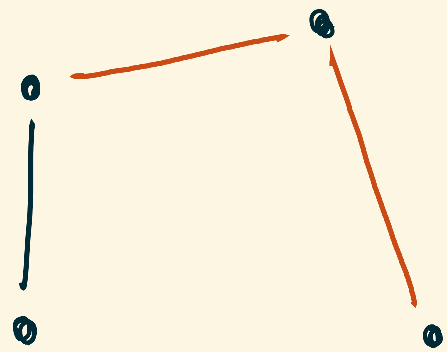
[BDL] Taking $\beta = \sigma_X$ for X spherical :

$$\lim_{n\to\infty} m_{\beta^n \tau, q}(Y) = q\text{-}\dim \mathrm{Hom}(X, Y)$$

upto simultaneous scalar

Limiting operations on Stab $\mathcal{C}$

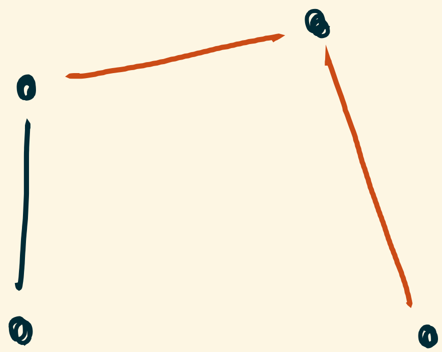
②



Shrink all but one of the simple semistables to zero.

Limiting operations on Stab $\mathcal{C}$

2



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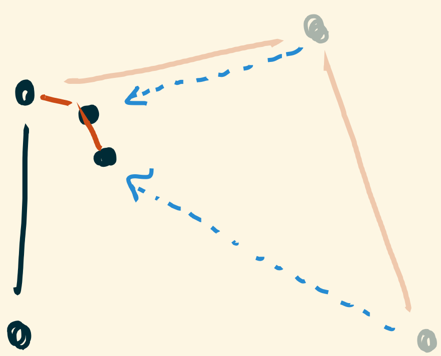

In the limit, the q-mass counts the "q-occurrences" of the remaining semistable in any given object.

Limiting operations on Stab $\mathscr{C}$

Moral : Limits may not make sense as stability conditions, but their q-masses make sense.

Limiting operations on $\mathrm{Stab}\,\mathcal{C}$

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Mass map

$$\begin{aligned} \mathrm{Stab}\,\mathcal{C} &\longrightarrow \mathbb{P}\mathbb{R}^{S} \\ \tau &\longmapsto [x \mapsto m_{q,\tau}(x)]/\sim \end{aligned}$$

Mass map & compactification

- [BDL, BBL] The mass map is injective, and $\overline{\mathrm{Stab}}^q\mathcal{C}$ is compact.

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- [BDL, BBL] The mass map is injective, and $\overline{\mathrm{Stab}}^{q}\mathcal{C}$ is compact.

- In the boundary, we see:

$\overline{\mathrm{hom}} := \lim\limits_{n \to \infty} m_{\beta^n \tau, q}$ for β = spherical twist

occ := q-occurrences of a fixed semistable

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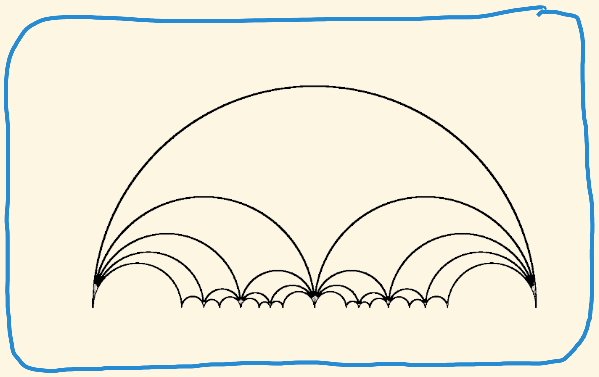
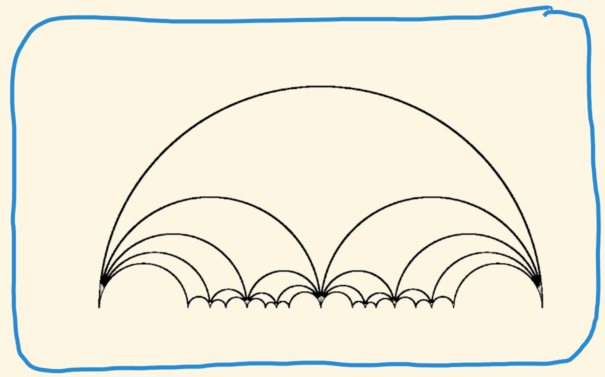
The story of the 3-strand braid group



The story of the 3-strand braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$



$$B_3 \longrightarrow PSL_2(\mathbb{Z})$$

$$\sigma_1 \longmapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \longmapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

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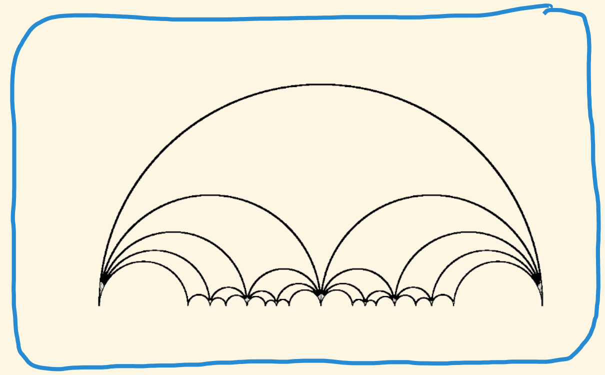


$$B_3 \longrightarrow PSL_2(\mathbb{Z})$$

$$\sigma_1 \longmapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \longmapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- $PSL_2(\mathbb{Z})$, and hence B_3, acts on $\mathbb{C} \cup \{\infty\}$ by fractional linear transformations
- Action preserves $\mathbb{H}$ and $\mathbb{R} \cup \{\infty\}$

The story of the 3-strand braid group

For the remainder of the talk, take

$$\mathcal{C} = \mathcal{C}(\bullet - \bullet) = \langle P_1, P_2 \rangle \circlearrowleft B_3$$

Fact:

$$\begin{array}{ccc} \mathrm{Stab}\,\mathcal{C} & \simeq & \mathbb{H} \\ \circlearrowright & & \circlearrowleft \\ B_3 & & B_3 \text{ via } PSL_2(\mathbb{Z}) \end{array}$$

The story of the 3-strand braid group

Take $\mathcal{C} = \mathcal{C}(\bullet - \bullet) = \langle P_1, P_2 \rangle \circlearrowright B_3$

Thm [B.-Deopurkar-Licata] : For $q=1$:

① $\overline{\hom}$ and occ coincide.

② $\overline{\hom}_X \mapsto \pm \dfrac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$ is a

B_3-equivariant bijection from the spherical objects of $\mathcal{C}$ to $\mathbb{Q} \cup \{\infty\}$

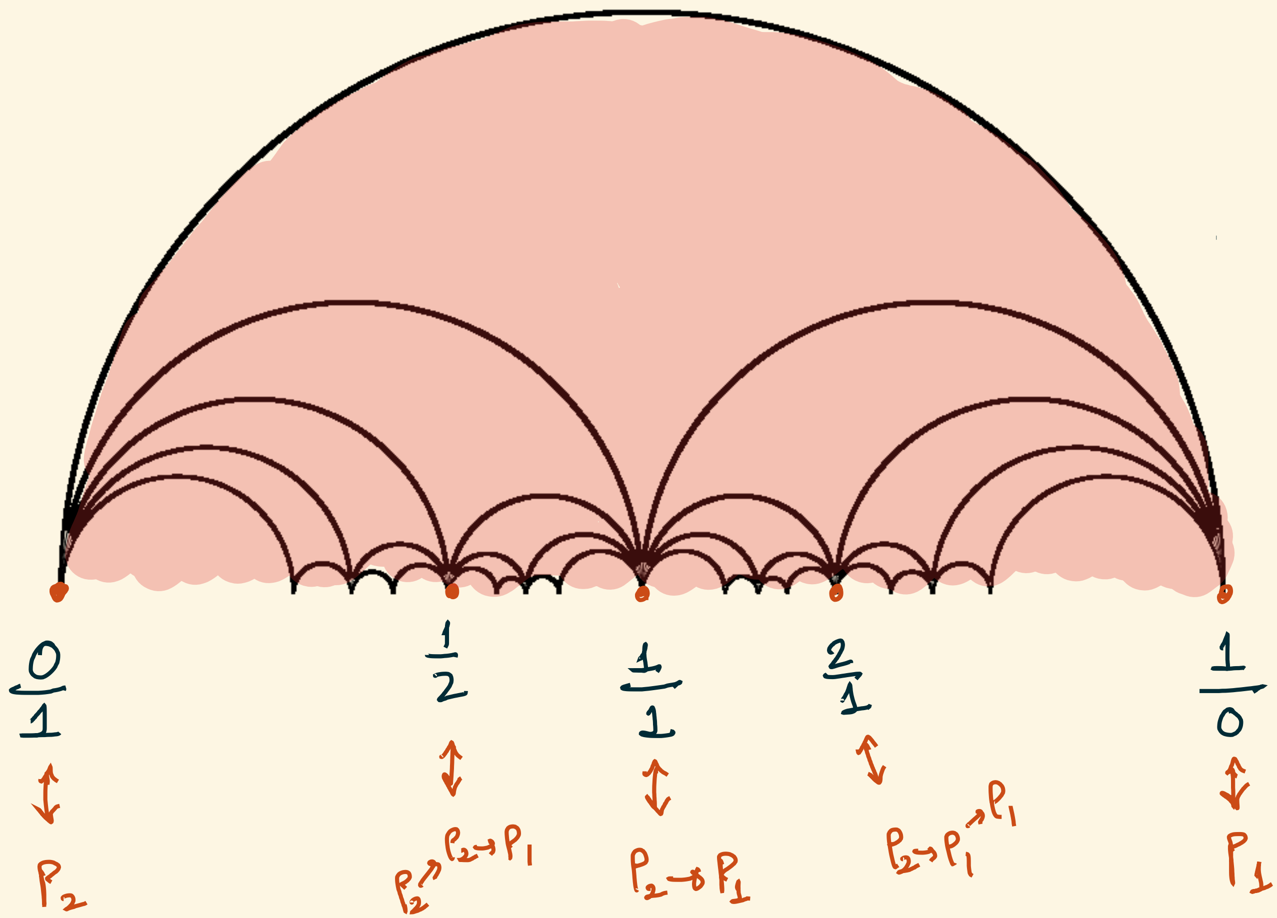
The $\overline{\hom}$ functionals as rationals

At $q=1$: The rationals can be recovered as the quotients

$$\pm \frac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$$ as X ranges over the spherical objects of $\mathcal{C}$.

The lion functionals as rationals

Pictorially, at $q=1$:



The q-deformed story for B_3

Question: Can we recover the q-rationals via some deformation of the quotients $\pm \dfrac{\hom(X, P_2)}{\hom(X, P_1)}$?

Answer: Yes, and more!

The q-deformed story for B_3

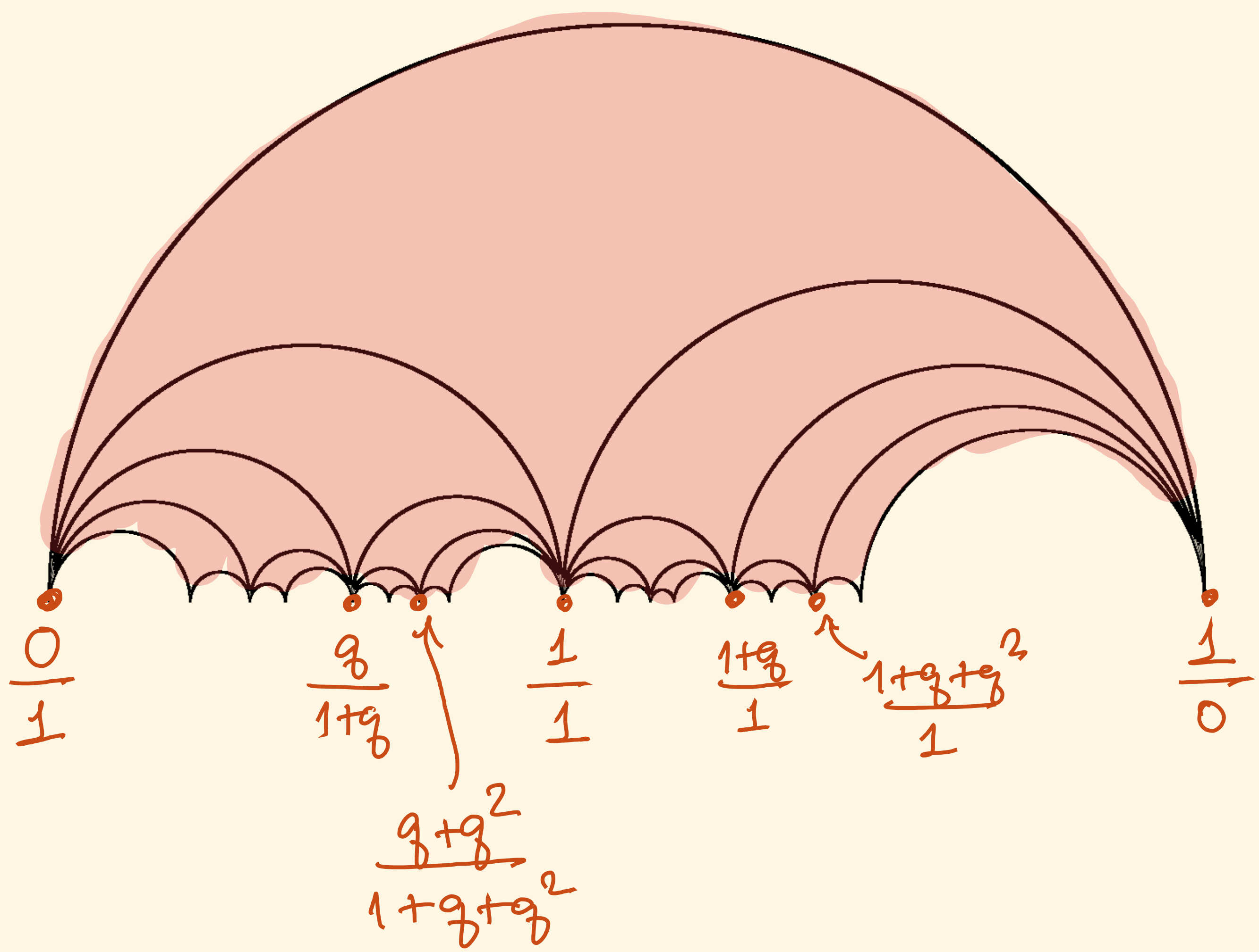
Thm [B.-Becker-Licata]

① $\pm q^{()} \dfrac{\operatorname{occ}(P_2, X)}{\operatorname{occ}(P_1, X)}$ are exactly the classical (right) q-deformed rationals of [M-G O]

② $\pm q^{()} \dfrac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$ is a new q-deformation of $\mathbb{Q}$. These are exactly the left q-rationals.

The q-deformed story for B_3

The right q-rationals at $q \neq 1$:



The q-deformed story for B_3

Thm [cont'd]

(3) $\overline{\hom}_X \mapsto \pm q^{()} \dfrac{\overline{\hom}_q(X, P_2)}{\overline{\hom}_q(X, P_1)}$ and

$\mathrm{occ}_X \mapsto \pm q^{()} \dfrac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$ are B_3-equivariant.

The q-deformed story for B_3

Thm [cont'd]

(3) $\overline{\hom}_X \mapsto \pm q^{()} \dfrac{\overline{\hom}_q(X, P_2)}{\overline{\hom}_q(X, P_1)}$ and

$\mathrm{occ}_X \mapsto \pm q^{()} \dfrac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$ are B_3-equivariant.

The B_3-action on the right is by fractional linear transformations via deformed B_3 matrices.

The q-deformed story for B_3

Upshot: for $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$, we have

① $$\left[\frac{r}{s}\right]_q^{\#} = \pm q^{()} \frac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$$ right q-deformed rational

② $$\left[\frac{r}{s}\right]_q^{\flat} = \pm q^{()} \frac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$$ left q-deformed rational.

New developments

For related recent work, see, e.g

* Fan-Qiu paper: gives model for q-rationals using bigraded intersections of curves

* Work of X. Ren, A. Thomas: further interpretations of q-numbers

* Leclere, Morier-Genoud, Ovsienko, et al: work on constructing & interpreting more general q-numbers.

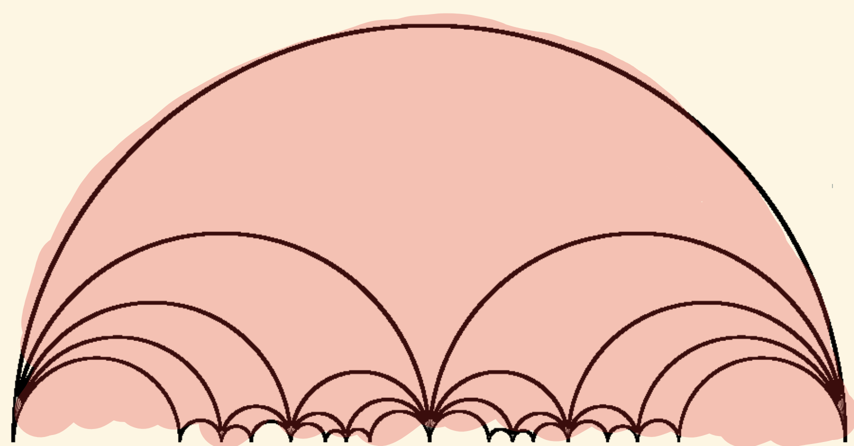
Specialising q

Take $q = 1$.

Consider the ideal triangle with vertices $0, 1, \infty$.

[corresponds to a piece of stability space]

The $PSL_2(\mathbb{Z})$-orbit:



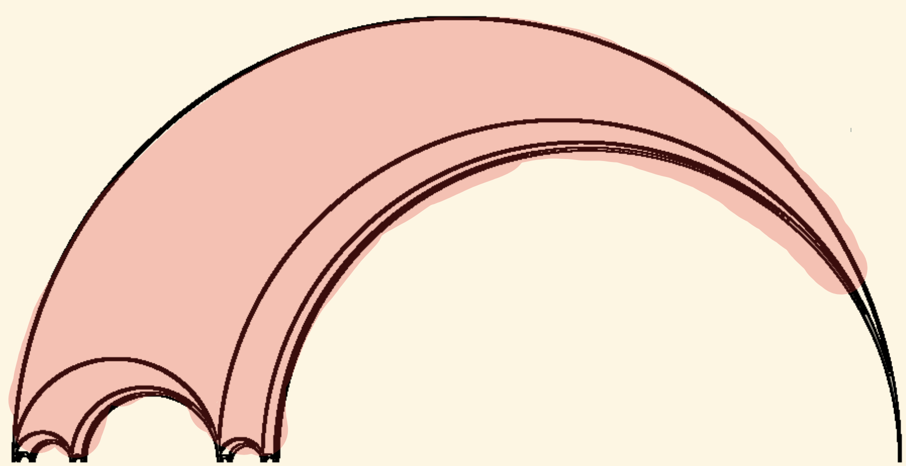
[$q = 1$]

Specialising q

Now fix $0 < q < 1$.

Consider the ideal triangle with vertices $0, 1, \infty$.

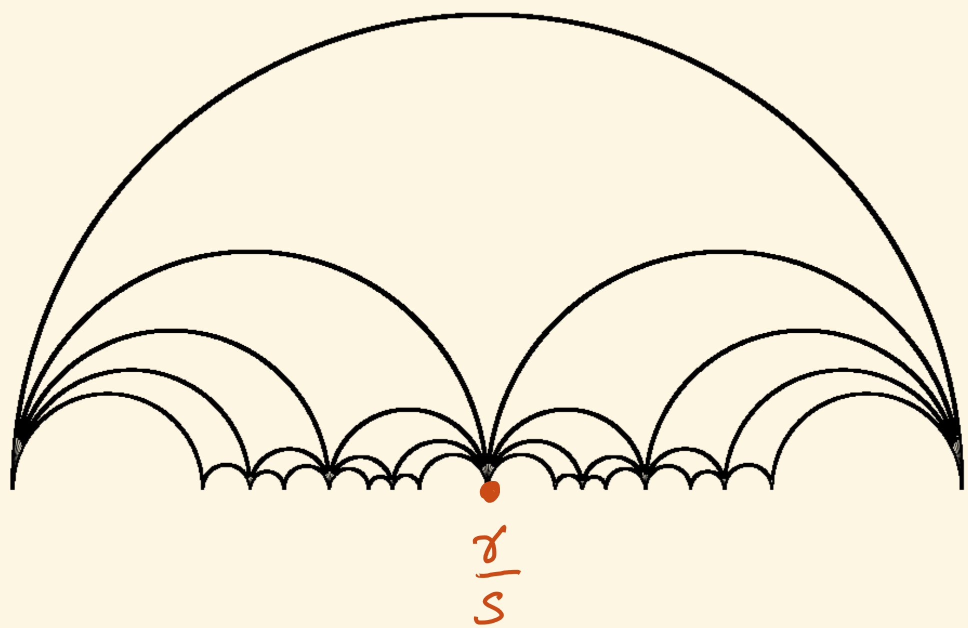
The $PSL_{2,q}(\mathbb{Z})$-orbit:



$[q = 0.3]$

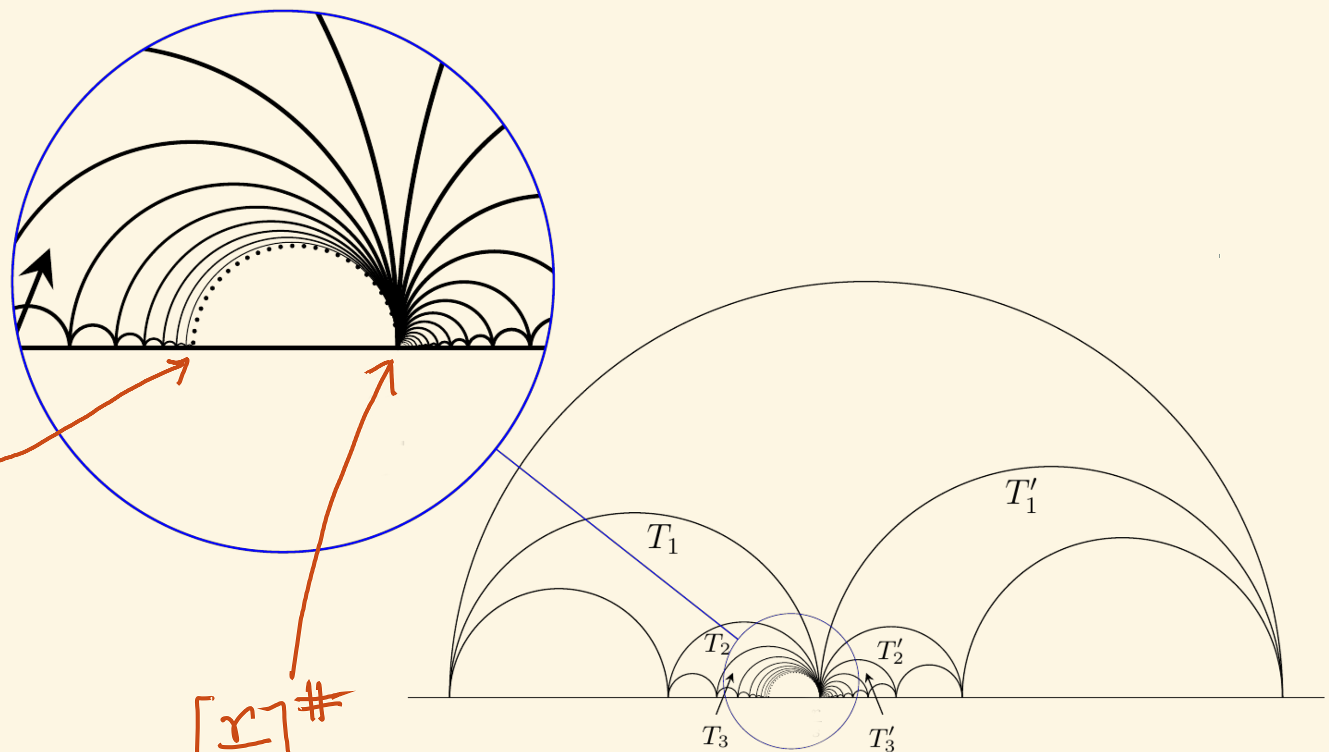
Specialising q

At $q=1$, left & right limits of Farey triangles agree.



Specialising q

At $q \neq 1$, the left & right limits of Farey triangles do not agree — we get $\left[\frac{r}{s}\right]_q^\flat$ & $\left[\frac{r}{s}\right]_q^\sharp$!



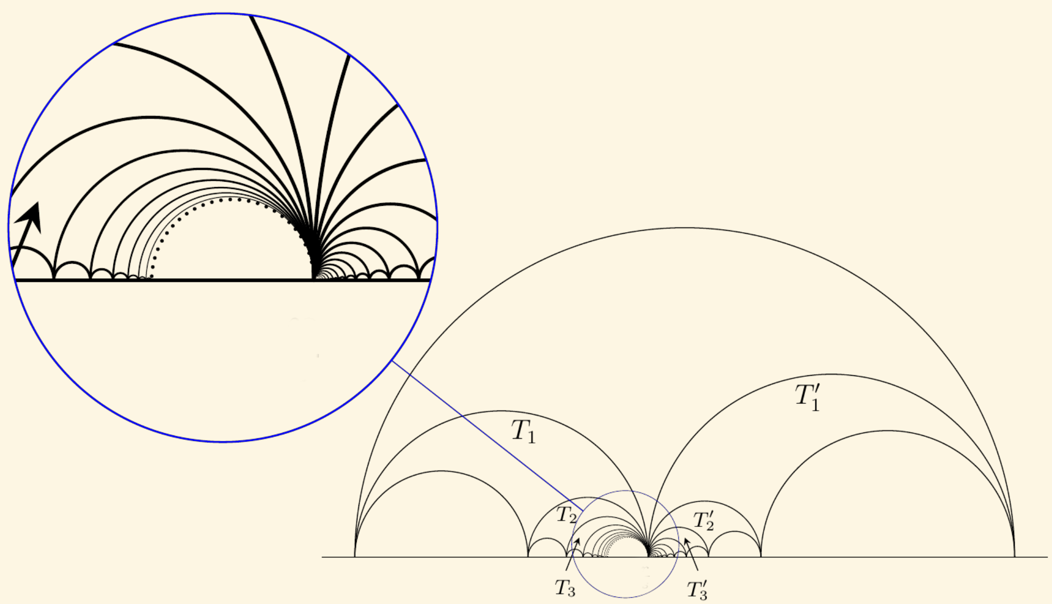
$\left[\frac{r}{s}\right]_q^\flat$

$\left[\frac{r}{s}\right]_q^\sharp$

Specialising q

At $q \neq 1$, the left & right limits of Farey triangles do not agree – we get $\left[\frac{r}{s}\right]_q^\flat$ & $\left[\frac{r}{s}\right]_q^\sharp$!

Moreover, the entire semicircle connecting them lies in the limit.



$\overline{\mathrm{Stab}}^q\, \mathcal{C}$ at a fixed positive q

Thm [B-Becker-Licata]

① The union of the closed semicircles $\left[\left[\frac{r}{s}\right]_q^{\flat}, \left[\frac{r}{s}\right]_q^{\sharp} \right]$ is dense in the boundary of $\overline{\mathrm{Stab}}^q\, \mathcal{C}$

② The remaining points of the boundary are exactly the "q-irrationals".

③ The boundary is homeomorphic to S^1.

Further questions

- Categorical interpretation of q-irrationals?
- Categorical interpretation of combinatorial properties of left & right q-rationals?
- Is the compactification actually a closed ball?
- Output from other categories?

Thank you!