A combinatorial study of Harder–Narasimhan filtrations (Sydney, 24 May 2024)

Broad motivation: Jordan–Hölder filtrations in abelian categories

Let $\mathcal{A}$ be a (finite-length) abelian category,

e.g. A-mod for some nice associative algebra

or e.g. category $\mathcal{O}_0$ for $\mathfrak{sl}_n$ etc.

$X \in \mathcal{A}$ has a Jordan–Hölder series w/ simple factors
(can coarsen to semisimple factors)

& JH multiplicities are well-defined

Do these have nice structure? $\longrightarrow$ YES (Kazhdan–Lusztig theory, e.g.)
very deep!

Idea: Study similar questions for Harder–Narasimhan factors of objects under a given stability condition.

Given an $\left\{\begin{array}{l}\text{abelian/}\\ \text{triangulated}\end{array}\right\}$ category, a (Bridgeland/GIT) stability condition is extra data which outputs, for each $X \in \mathcal{C}$, a canonical finite filtration with semistable factors

(We'll go back & forth between abelian/Δted)

$$\begin{array}{ccccc}
\mathcal{A} & \xleftrightarrow[\text{data}]{\text{equivalent}} & D^b(\mathcal{A}) & & D^b(\mathcal{A}) + \text{stability condition} \\
\downarrow & & \downarrow & & \downarrow \\
\text{JH filtration} & & \text{JH + cohomology filtration} & \xrightarrow{\text{coarsen}} & \text{HN filtration}
\end{array}$$

HN multiplicities well-defined but also filtration well-defined!

②

$$\begin{tikzcd}
\mathcal{A} \arrow[r, squiggly, "\text{upgrade}"] \arrow[d, squiggly, "\text{coarsen}"'] & D^b(\mathcal{A}) \arrow[d, squiggly, "\text{coarsen}"] \\
\mathcal{A} + \text{stability function} \arrow[r, squiggly, "\text{upgrade}"] & \boxed{D^b(\mathcal{A}) + \text{stability condition}}
\end{tikzcd}$$

(JH filtration) — (cohomology + JH filtration)

(HN filtration) — (HN filtration)

MUCH more flexible!

<u>Broad question</u>: What can appear in the HN support of (nice) objects? What is the combinatorics?

<u>Specific case of interest</u>: 2CY Δ cat of connected graph (Usually, Γ = type A_n, or later D_4 }, but questions make sense for any Γ!)

$\Gamma^{dbl} = \bullet \rightleftarrows \bullet \rightleftarrows \bullet \rightleftarrows \bullet$; take $A_\Gamma = k\Gamma^{dbl}$/relations

where ~~A~~ relations $\equiv$

- $(i \to j \to k) = 0$ if $i \neq k$
- $(i \to j \to i) = (i \to k \to i)$;

$\mathcal{C} = K^b(\mathbb{Z}\text{-gr-proj})$ / (grading shift = homological shift)

$\mathcal{C}$ is 2CY : $\mathrm{Hom}^i_{\mathcal{C}}(A, B) \simeq \mathrm{Hom}^{2-i}_{\mathcal{C}}(B, A)^\vee$

$\mathcal{C}$ has ~~[illegible]~~ $P_i = A_\Gamma . e_i$, which are now simple in the standard heart $\langle P_i \rangle^{ab}$

std heart has finite length $\Rightarrow$ objects have JH filtrations.

$\mathrm{End}^\bullet(P_i) \cong k[\ell]/\ell^2$ where $\deg \ell = 2$

(these are <u>spherical</u>) $+$ $\mathrm{Hom}^m(P_i, P_j) = k$ iff $m = 1$

P_i gives rise to endofunctor $\sigma_{P_i} : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$

$\sigma_{P_i}(-) = [P_i \otimes_k {}_iP \xrightarrow{\mu} A_\Gamma] \otimes (-)$ [Rouquier-type complex of bimodules]

$\sigma_{P_i}^{-1}(-) = [A_\Gamma \xrightarrow{\text{co-}\mu} P_i \otimes {}_iP] \otimes (-)$
(+shift)

$\equiv$ spherical twist functor, sends sphericals $\mapsto$ sphericals

and $\langle \sigma_{P_i} \rangle$ ~~generate~~ satisfy the relations of the braid group B_Γ of Γ, which is ~~Braid type~~ B_{n+1} in type A_n.

<u>Idea</u>: Look at filtration factors of objects in $\mathcal{S}$ = spherical objects
specifically $\beta(P_i)$ for $\beta \in B_\Gamma$.
(Motivate this as we go.)

type A_2 : Rouquier-Zimmerman tried to write down JH multiplicities of $\beta(P_i) \longrightarrow$ succeeded, but it is messy.
$\hookrightarrow$ reproved generalized from the point of view of stability conditions
(B.-Deopurkar-Licata) + (B.-Becker-Licata)

<u>Ans</u>: A finite state machine governs everything.

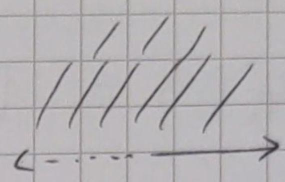
$$\begin{tikzcd}
\sigma_1 \circlearrowright \left[P_1, \binom{P_1}{P_2}\right] \arrow[r, shift left, "\sigma_2^{-1}"] \arrow[d, shift right, "\sigma_2"'] & \left[\binom{P_1}{P_2}, P_2\right] \circlearrowleft \sigma_2^{-1} \arrow[l, shift left, "\sigma_1"] \arrow[d, shift right, "\sigma_1^{-1}"'] \\
\hat{\sigma}_1 \circlearrowright \left[\binom{P_2}{P_1}, P_1\right] \arrow[r, shift left, "\sigma_2"] \arrow[u, shift right, "\sigma_2^{-1}"'] & \left[P_2, \binom{P_2}{P_1}\right] \circlearrowleft \sigma_2 \arrow[l, shift left, "\sigma_1^{-1}"] \arrow[u, shift right, "\sigma_1"']
\end{tikzcd}$$

[BBL]

Factors appearing are not simple/semisimple
→ but they are semistable!

Crash course on stability functions

$\mathcal{A}$ abelian
$= \langle P_1, \dots, P_n \rangle^{ab}$

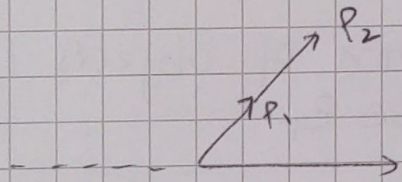
Consider additive fn $K_0(\mathcal{A}) \xrightarrow{Z} \mathbb{H} =$ 

(determined by values on P_i)

$X \in \mathcal{A}$ is <u>semistable</u> if $\forall\ 0 \subsetneq Y \subsetneq X$,

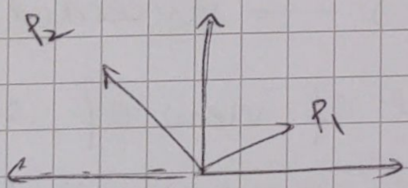
~~E.g.~~ $\arg(Z(Y)) \le \arg Z(X)$

<u>E.g.</u> (A_2): $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ & $\begin{pmatrix} P_2 \\ P_1 \end{pmatrix}$ semistable

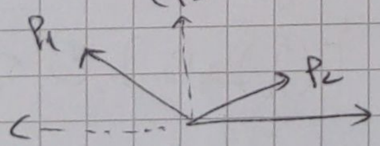


$\begin{pmatrix} P_2 \\ P_1 \end{pmatrix}$ semistable

automaton was in this case.

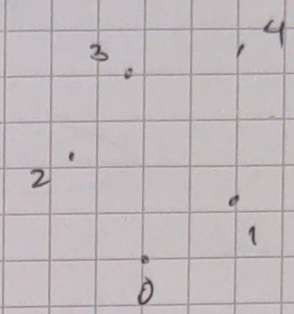


$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ semistable



<u>More generally</u>: (type A_n) (Khovanov-Seidel, Thomas '06 + B-D-L in preparation)

stability condition ~~("standard")~~ $\longleftrightarrow$ configuration of $(n+1)$ points



(label bottom to top)

$P_i \longleftrightarrow i - i+1$

semistable $\longleftrightarrow$ straight / almost straight segment

spherical $\longleftrightarrow$ curve / isotopy.

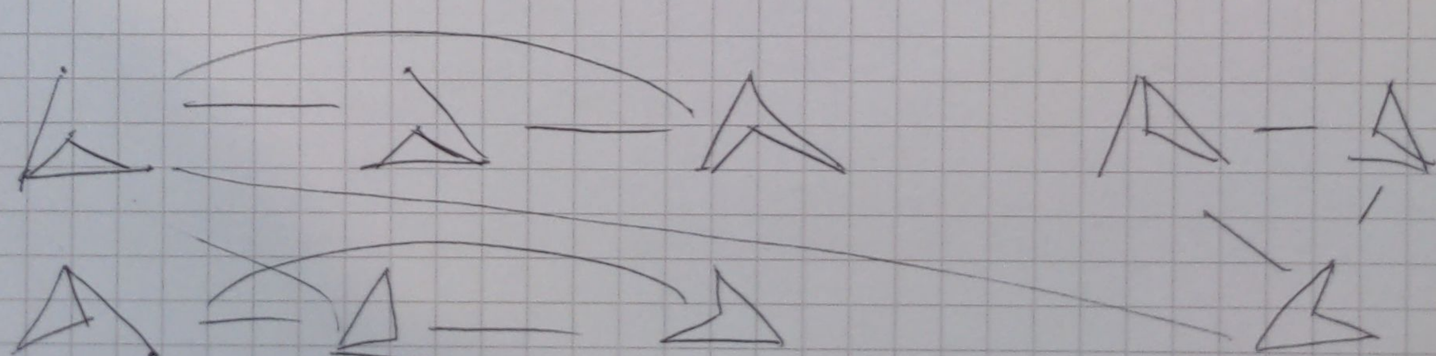
~~Observation: Given~~

HN support $\longleftrightarrow$ pull tight!

Constraints on HN support: (subset of semistables)

1. No crossings = planar drawing
2. No tripods = "pointedness"
3. Miss an outside edge.

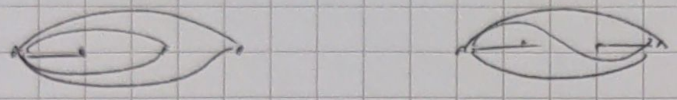
E.g.



(6)

Rote-Santos-Streinu : polytope of pointed pseudo-triangulations for pts in generic position

B. - Pilaud (in progress) : polytope of pointed pseudo-Δ for pts in maximally special position



etc

Other types $\to$ "ppt" combinatorics
(B. - Deopurkar - Licata - Queffelec) in preparation : type D_4 !