FAITHFULNESS QUESTIONS ABOUT GENERALISED BURAU REPRESENTATIONS

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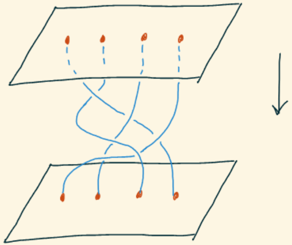
Available at https://arxiv.org/abs/2409.00144

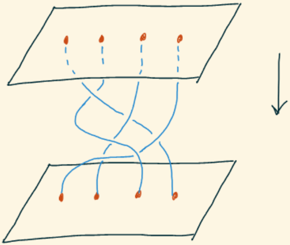
THE BRAID GROUP B_n

- A braid is a continuous motion of n distinct points in the plane that
  - avoids collisions; and
  - ends at the original configuration.



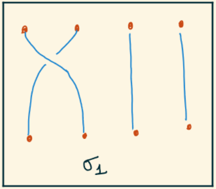
THE BRAID GROUP B_n

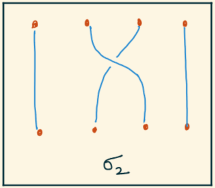
- We consider motions up to homotopy.
- Nevertheless, the braid group is infinite.
- The symmetric group S_n is a natural quotient.



THE BRAID GROUP B_n

- B_n is generated by $\{\sigma_i \mid 1 \leq i < n\}$.
- We have the relations:
  - $\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| > 1$;
  - $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad |i - j| = 1$.





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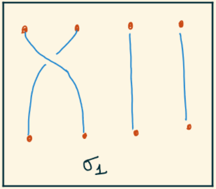

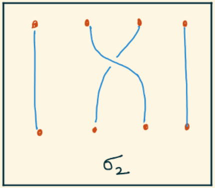

REPRESENTATIONS OR MATRIX REALISATIONS OF B_n

Does B_n embed into a finite matrix group?

Yes (early 2000s): B_n has a faithful representation of dimension $\binom{n}{2}$.

Are there smaller candidates?

The Burau representation was a good candidate for a while. It has dimension $(n-1)$.

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HISTORY OF THE BURAU REPRESENTATION

- The Burau representation was defined in the 1930s [Burau].
- It is faithful for $n \leq 3$ [Magnus–Peluso ’60s].
- It is not faithful for $n \geq 5$ [Moody ’91, Long–Paton ’93, Bigelow ’99]
- The $n = 4$ case is still unknown!

THE BURAU REPRESENTATION OF B_n

- The Burau action is on a vector space with basis $\{\alpha_i \mid 1 \leq i < n\}$, equipped with a certain q-pairing $\langle -, - \rangle$.
- The formula is:

$$\sigma_i(v) = v - \langle \alpha_i, v \rangle \alpha_i.$$

- It deforms the standard or reflection representation of S_n.

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ARTIN–TITS GROUPS

- Let Γ be an undirected graph.
- The group B_Γ is generated by $\{\sigma_i \mid i \in \Gamma\}$.
- We have the relations:
  - $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $i \not\leftrightarrow j$;
  - $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ if $i \leftrightarrow j$.

$\sigma_1 \sigma_3 = \sigma_3 \sigma_1$

$\sigma_4 \sigma_2 \sigma_4 = \sigma_2 \sigma_4 \sigma_2$

$\vdots$

B_n AS AN ARTIN–TITS GROUP

- B_n is the Artin–Tits group of a chain with $(n-1)$ vertices.
- For instance, the following graph gives $B_\Gamma = B_4$.

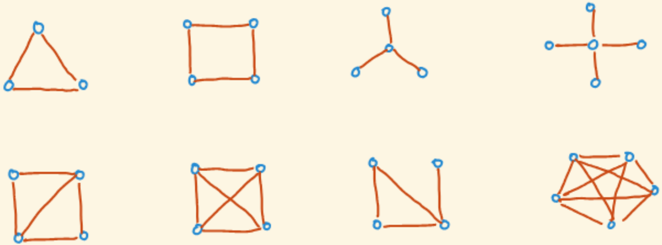
THE BURAU REPRESENTATION FOR ARTIN–TITS GROUPS

- As before, the Burau action is on a vector space spanned by $\{\alpha_i \mid i \in \Gamma\}$, with a pairing $\langle -, - \rangle$.
- The action is by the same formula:

$$\sigma_i(v) = v - \langle \alpha_i, v \rangle \alpha_i.$$

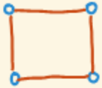
FAITHFUNLESS QUESTIONS

- The Burau representation of B_n is unfaithful for $n \geq 5$.
- So if Γ contains a 4-chain as a full subgraph, then the Burau representation of B_Γ is unfaithful.
- However, we are left with a large number of open cases!



RESULTS

Theorem (B.–Queffelec)
The Burau representation of the Artin–Tits group of the square (type $\widehat{A}_3$) is not faithful.



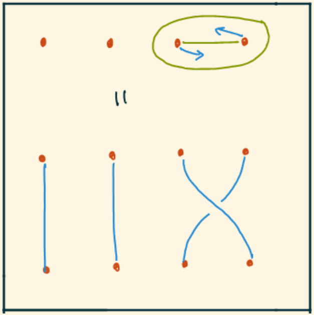
Theorem (B.–Queffelec)
The Burau representation of the Artin–Tits group of the 3-pointed star (type D_4) is not faithful over $\mathbb{Z}/n\mathbb{Z}$ for $n \leq 16$.

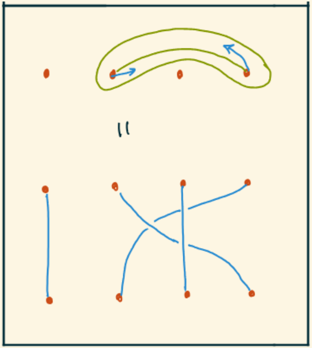
BROAD STRATEGY

- Consider a richer action of the group that remembers more information.
- However, this action may not be linear.

INTERLUDE: ARCS GIVE RISE TO BRAIDS

A half Dehn twist around an arc joining two points is the braid that swaps the endpoints "counterclockwise" around this arc.

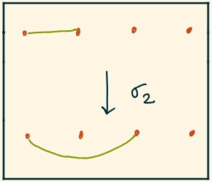


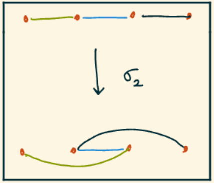


INTERLUDE: B_n ACTS ON ARCS

- σ_i moves an arc by twisting around the $(i, i+1)$ segment.
- B_n also acts on collections of arcs.
- There is a correspondence

$$\text{arcs} \rightsquigarrow \text{Burau vectors};$$
$$\text{intersection numbers} \rightsquigarrow \text{pairing}.$$





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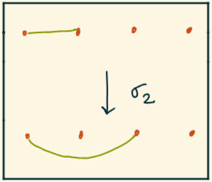

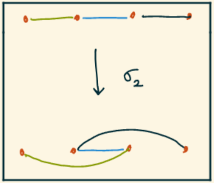

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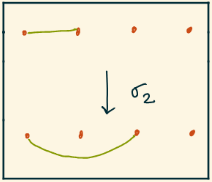
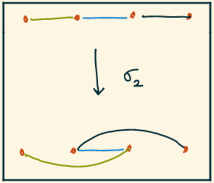
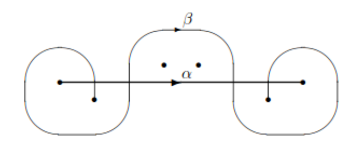


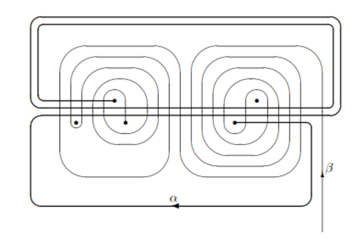


STRATEGY TO ELEMENTS IN THE KERNEL OF BURAU

Strategy for the classical braid group (Bigelow)

If arcs α, β intersect but corresponding vectors have zero pairing, then $\alpha\beta\alpha^{-1}\beta^{-1}$ is in the kernel.





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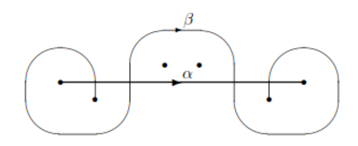

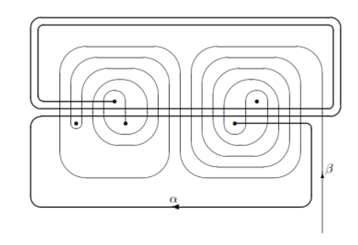

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For Artin–Tits groups, we don't usually have any action on arcs!

Key insight in general

- Instead, consider an action on a triangulated category.
- Instead of counting intersections between arcs, count the number of morphisms between objects in the category.

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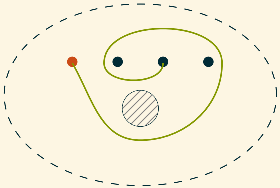
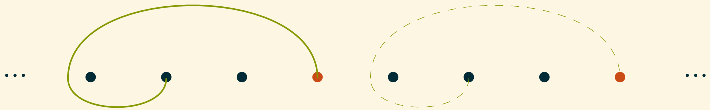
ARC ACTION FOR THE SQUARE GRAPH $\widetilde{A}_3$

- $\widetilde{A}_3$ is realised by arcs on the annulus with 4 marked points.

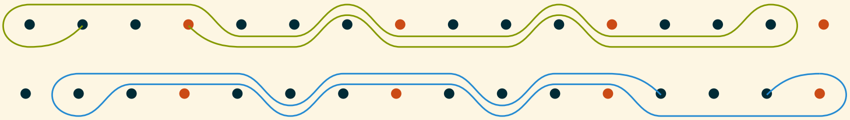


- We can unfold an arc to the universal cover of the annulus.



KERNEL ELEMENT IN BURAU OF $\widetilde{A}_3$

The following two arcs intersect, but have zero pairing.



Conclusion

$$\alpha = \sigma_3^2\sigma_4\sigma_3\sigma_2\sigma_1\sigma_3^{-1}\sigma_4\sigma_3\sigma_2\sigma_1^{-2}\sigma_4\sigma_3\sigma_4^{-1}\sigma_1^2\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-2},$$

$$\beta = \sigma_1^2\sigma_2^{-1}\sigma_4\sigma_1\sigma_3^{-1}\sigma_2\sigma_4^{-1}\sigma_3\sigma_1\sigma_4\sigma_1\sigma_2^{-1}\sigma_4^{-2}\sigma_3\sigma_2\sigma_3^{-1}\sigma_4^2\sigma_2\sigma_1^{-1}\sigma_4^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_4 \sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_4^{-1}\sigma_2\sigma_1^{-2}.$$

Then $\alpha\beta\alpha^{-1}\beta^{-1}$ is a non-trivial element in the kernel.

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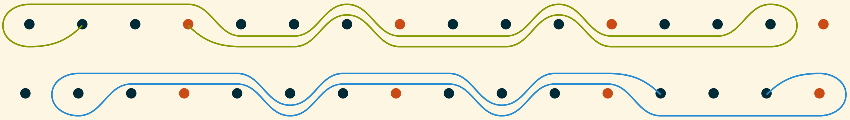

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HOW DO WE FIND SUCH EXAMPLES?

- Targeted computer search guided by several heuristics.
- For $\widetilde{A}_3$, easier to use a group action on a category than on arcs!
- In any case, the categorical approach is the one that generalises to D_4 and other cases.

THANK YOU!