q-DEFORMED RATIONAL NUMBERS VIA CATEGORICAL GROUP ACTIONS

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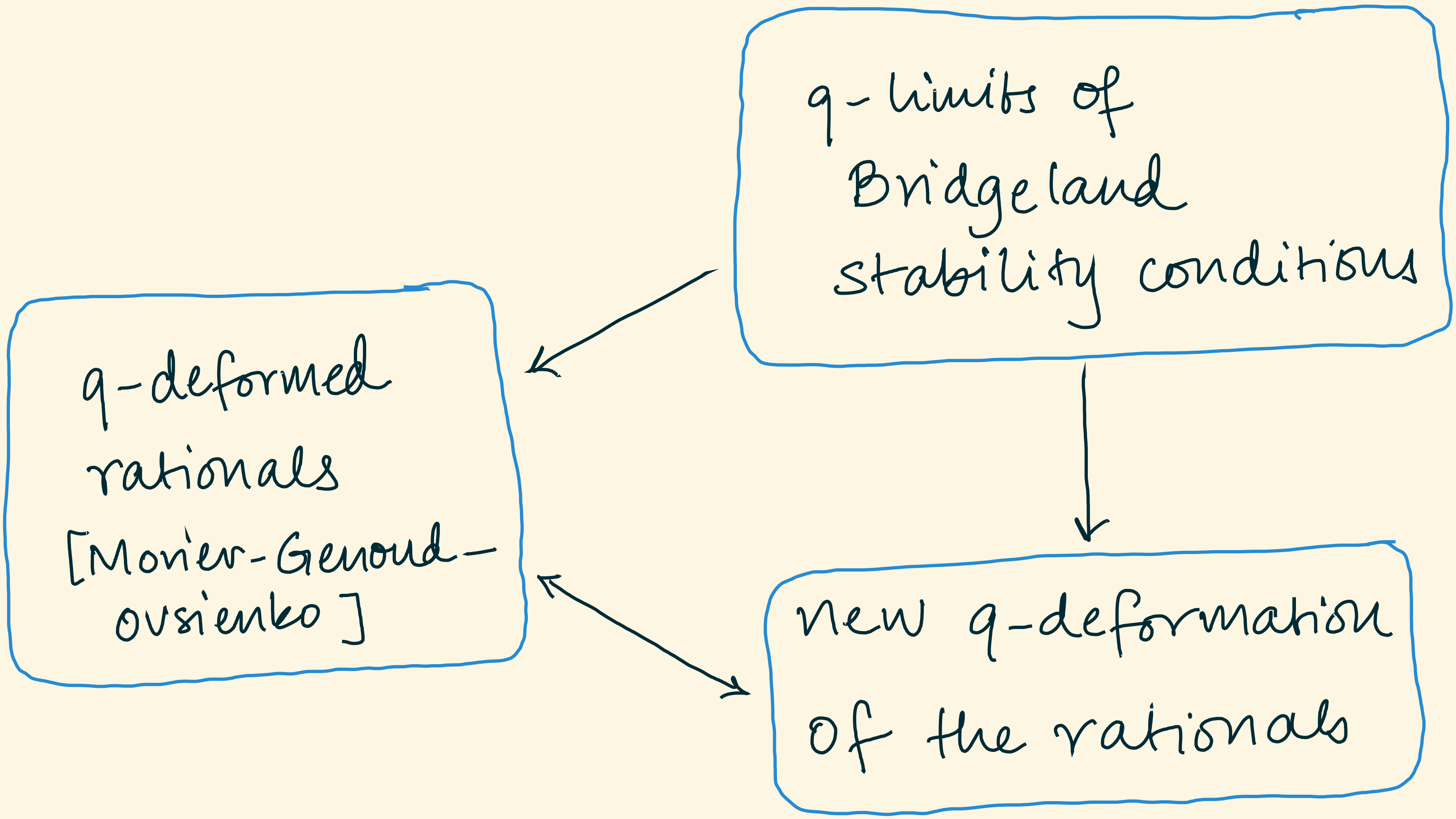
Outline


q-limits of Bridgeland stability conditions
q-deformed rationals [Morier-Genoud-Ovsienko]
new q-deformation of the rationals

Outline

q-limits of Bridgeland stability conditions

q-deformed rationals [Morier-Genoud-Ovsienko]

new q-deformation of the rationals

Fractional linear action of B_3

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

There is a homomorphism

$$B_3 \to PSL_2(\mathbb{Z}) :$$

$$\sigma_1 \mapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \mapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Fractional linear action of B_3

$PSL_2(\mathbb{Z})$ acts on $\mathbb{R} \cup \{\infty\}$ via fractional linear transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \left(\frac{r}{s}\right) := \frac{ar+bs}{cr+ds}$$

$\Rightarrow$

- B_3 acts on $\mathbb{R} \cup \{\infty\}$.
- The action preserves $\mathbb{Q} \cup \{\infty\}$.

Fractional linear action of B_3

Can be realised via continued fractions.

Let $$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots\, a_{2n}}}$$

Then

$$\boxed{\frac{r}{s} = \sigma_1^{-a_1} \sigma_2^{a_2} \sigma_1^{-a_3} \sigma_2^{a_4} \cdots \sigma_1^{-a_{2n-1}} \sigma_2^{a_{2n}} (\infty)}$$

Classical (right) q-deformed rationals

Consider deformed matrices:

$$\sigma_{1,q} := \begin{bmatrix} q^{-1} & -q^{-1} \\ 0 & 1 \end{bmatrix}, \quad \sigma_{2,q} := \begin{bmatrix} 1 & 0 \\ 1 & q^{-1} \end{bmatrix}$$

These generate a copy of B_3 in

$$PSL_2(\mathbb{Z}[q^{\pm}]).$$

Classical (right) q-deformed rationals

Let $\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots a_{2n}}}$

The corresponding right q-deformation is:

$$\boxed{\left[\frac{r}{s}\right]^{\#}_{q} = \sigma_{1,q}^{-a_1}\, \sigma_{2,q}^{a_2} \cdots \sigma_{1,q}^{-a_{2n-1}}\, \sigma_{2,q}^{a_{2n}} (\infty).}$$

[Morier-Genoud – Ovsienko]

Left q-deformed rationals

Let $\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots a_{2n}}}$

The corresponding left q-deformation is:

$$\boxed{\left[\frac{r}{s}\right]^{b}_{q} = \sigma_{1,q}^{-a_1}\, \sigma_{2,q}^{a_2} \cdots\cdots \sigma_{1,q}^{-a_{2n-1}}\, \sigma_{2,q}^{a_{2n}} \left(\frac{1}{1-q}\right).}$$

Examples

$$\left[\frac{1}{2}\right]_q^\flat = \frac{q^2}{1+q^2} \qquad \left[\frac{1}{2}\right]_q^\sharp = \frac{q}{1+q}$$

$$\left[\frac{5}{2}\right]_q^\flat = \frac{1+q+q^2+q^3+q^4}{1+q^2} \qquad \left[\frac{5}{2}\right]_q^\sharp = \frac{1+2q+q^2+q^3}{1+q}$$

$$\left[\frac{5}{3}\right]_q^\sharp = \frac{1+q+q^2+q^3+q^4}{1+q+q^3} \qquad \left[\frac{5}{3}\right]_q^\sharp = \frac{1+q+2q^2+q^3}{1+q+q^2}$$

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Categorical B_3- action

$$\begin{tikzcd} B_3 \arrow[d, two heads] & \sigma_1 \arrow[d, mapsto] & \sigma_2 \arrow[d, mapsto] \\ S_3 & (1\,2) & (2\,3) \end{tikzcd}$$

symmetric group

Categorical B_3-action

$$
\begin{array}{ccc}
B_3 & & \\
\big\downarrow & & \\
S_3 & \circlearrowright & \mathbb{C}^2
\end{array}
$$

symmetric group

standard representation

$$= \frac{(\mathbb{C}^3 \text{ via permutations})}{\operatorname{span}((1,1,1))}$$

Categorical B_3-action

(triangulated category)

$$\begin{tikzcd} B_3 \arrow[d, two heads] & \circlearrowright & \mathcal{C} \arrow[d, squiggly, "\text{Grothendieck group} \otimes \mathbb{C}"] \\ S_3 & \circlearrowright & \mathbb{C}^2 \end{tikzcd}$$

symmetric group

standard representation

$$= \frac{(\mathbb{C}^3 \text{ via permutations})}{\text{span}((1,1,1))}$$

Categorical B_3-action

$S_3 \circlearrowright \mathbb{C}^2$

$\mathbb{C}^2 = \langle v_1, v_2 \rangle$

basis vectors

vs

$B_3 \circlearrowright \mathcal{C}$

$\mathcal{C} = \langle P_1, P_2 \rangle$

generating objects

Categorical B_3-action

$S_3 \circlearrowright \mathbb{C}^2$ vs $B_3 \circlearrowright \mathcal{C}$

$\mathbb{C}^2 = \langle v_1, v_2 \rangle$

$\sigma_1(v_1) = -v_1$

$\sigma_1(v_2) = v_1 + v_2$

Euler char

$\mathcal{C} = \langle P_1, P_2 \rangle$

$$\begin{cases} \sigma_1(P_1) = P_1[1] \\ \sigma_1(P_2) = P_1 \to P_2 \end{cases}$$

complexes

Categorical B_3-action

$S_3 \circlearrowright \mathbb{C}^2$ vs $B_3 \circlearrowright \mathcal{C}$

$\mathbb{C}^2 = \langle v_1, v_2 \rangle$

$\sigma_1(v_1) = -v_1$

$\sigma_1(v_2) = v_1 + v_2$

$$\langle v_i, v_j \rangle = \begin{cases} -1 & i \neq j \\ 2 & i = j \end{cases}$$

$\mathcal{C} = \langle P_1, P_2 \rangle$

$\sigma_1(P_1) = P_1[1]$

$\sigma_1(P_2) = P_1 \to P_2$

$$\mathrm{Hom}_{\mathcal{C}}(P_i, P_j) = \begin{cases} \mathbb{C}\langle 1 \rangle & i \neq j \\ \mathbb{C} \oplus \mathbb{C}\langle 2 \rangle & i = j \end{cases}$$

Categorical B_3-action

The objects P_1 & P_2 are "spherical":

$$\mathrm{Hom}_{\mathcal{C}}(P_i, P_i) = \mathbb{C} \oplus \mathbb{C}\langle 2 \rangle$$

Any spherical object X defines autoequivalence $\sigma_X : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$.

In particular, σ_{P_i} = action of $\sigma_i \in B_3$

Categorical B_3-action

The functors σ_{P_1} & σ_{P_2} braid:

$\sigma_{P_1}\sigma_{P_2}\sigma_{P_1} \simeq \sigma_{P_2}\sigma_{P_1}\sigma_{P_2}$ as desired.

More generally, if $\beta \in B_3$, then

- $\beta(P_i)$ is spherical
- $\sigma_{\beta(P_i)}$ = action of $\beta\sigma_i\beta^{-1}$.

Bridgeland stability conditions & B_3- action

We will encounter q-rationals again by taking "limiting q-sizes" of objects in $\mathcal{C}$.

These are provided by Bridgeland stability conditions.

Bridgeland stability conditions & B_3- action

Choosing a stability condition on $\mathcal{C}$ is like choosing a basis of a vector space.

- The "basis objects" are called semistable.
- Each object X has a canonical expression as a composition of semistables.

Bridgeland stability conditions & B_3- action

Choosing a stability condition on $\mathcal{C}$ is like choosing a basis of a vector space.

- The "basis objects" are called <u>semistable</u>.
- Each object X has a canonical expression as a composition of <u>semistables</u>.
- [Bridgeland] The moduli space of stability conditions is a complex manifold.

Bridgeland stability conditions & B_3- action

- Every semistable A has a
  - mass $\quad m(A) \in \mathbb{R}_{>0}$
  - phase $\quad \phi(A) \in \mathbb{R}$

- The "q-size" of any $X \in \mathcal{C}$ is

$$m_q(X) := \sum q^{\phi(A_i)} m(A_i)$$

(sum over all composition factors of X)

Bridgeland stability conditions & B_3-action

For our category, $\operatorname{Stab} \mathcal{C}$ is the complex upper half plane $\mathbb{H}$.

The induced action $B_3 \circlearrowright \mathbb{H}$ is by fractional linear transformations, via $B_3 \to PSL_2(\mathbb{Z}) \circlearrowright \mathbb{H}$.

Limiting operations on $\operatorname{Stab}\mathcal{C}$

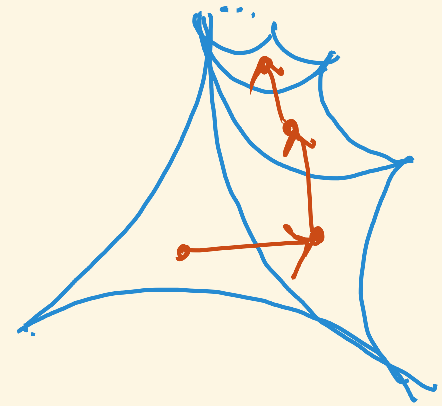
$\mathbb{H}$ is not compact, so we can't take limits in $\operatorname{Stab}\mathcal{C}$.

What if we take limits of the mass map? (Suitably interpreted)

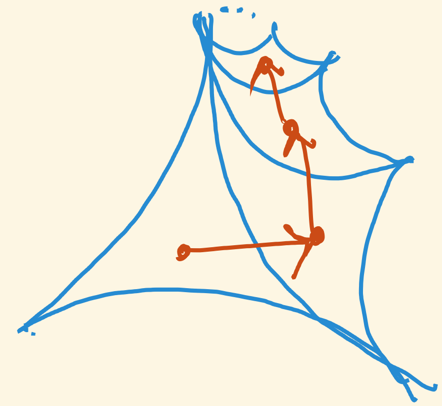
Limiting operations on Stab $\mathcal{C}$

① Fix $\beta \in B_r$ and $\tau \in \mathrm{Stab}\,\mathcal{C}$.

Consider $\lim\limits_{n \to \infty} \beta^n \tau$.



Limiting operations on Stab $\mathcal{C}$



① Fix $\beta \in B_\Gamma$ and $\tau \in \operatorname{Stab}\mathcal{C}$.

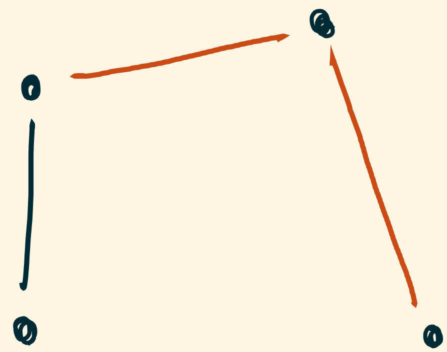
Consider $\lim_{n\to\infty} \beta^n \tau$.

Theorem: If $\beta = \sigma_X$ for X spherical:

$$\lim_{n\to\infty} m_q(Y, \beta^n\tau) = \left(q\text{-Euler char of } \operatorname{Hom}(X,Y)\right) \Big/ \text{scalar}$$

[B.–Deopurkar–Licata]

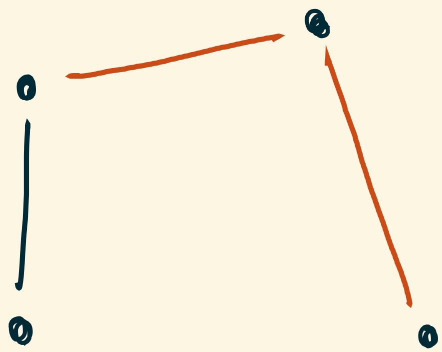
Limiting operations on Stab $\mathcal{C}$

② 

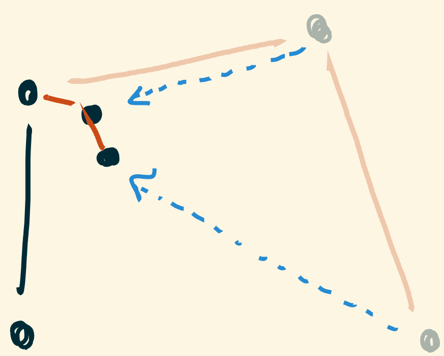
Shrink all but one of the simple semistables to zero.

Limiting operations on Stab $\mathcal{C}$

②



Shrink all but one of the simple semistables to zero.



In the limit, the q-mass counts the "q-occurences" of the remaining semistable in any given object.

Limiting operations on Stab $\mathcal{C}$

Moral: Limits may not make sense as stability conditions, but their q-masses make sense.

Limiting operations on $\mathrm{Stab}\,\mathcal{C}$

Moral : Limits may not make sense as stability conditions, but their q-masses make sense.

Mass map

$$\mathrm{Stab}\,\mathcal{C} \longrightarrow \mathbb{PR}^{S}$$

$$\tau \longmapsto [x \mapsto m_{q,\tau}(x)]/\sim$$

Mass map & compactification

- Theorem: The mass map is injective, and $\overline{\mathrm{Stab}}^{q}\mathcal{C}$ is compact. [BDL, B.-Becker-Licata]

Mass map & compactification

- Theorem: The mass map is injective, and $\overline{\mathrm{Stab}}^{q}\mathcal{C}$ is compact. [BDL, B.-Becker-Licata]

- In the boundary, we see:

$$\mathrm{hom} := \lim_{n\to\infty} m_{\beta^n\tau, q} \quad \text{for } \beta = \text{spherical twist}$$

occ := q-occurrences of a fixed semistable

The story at $q = 1$

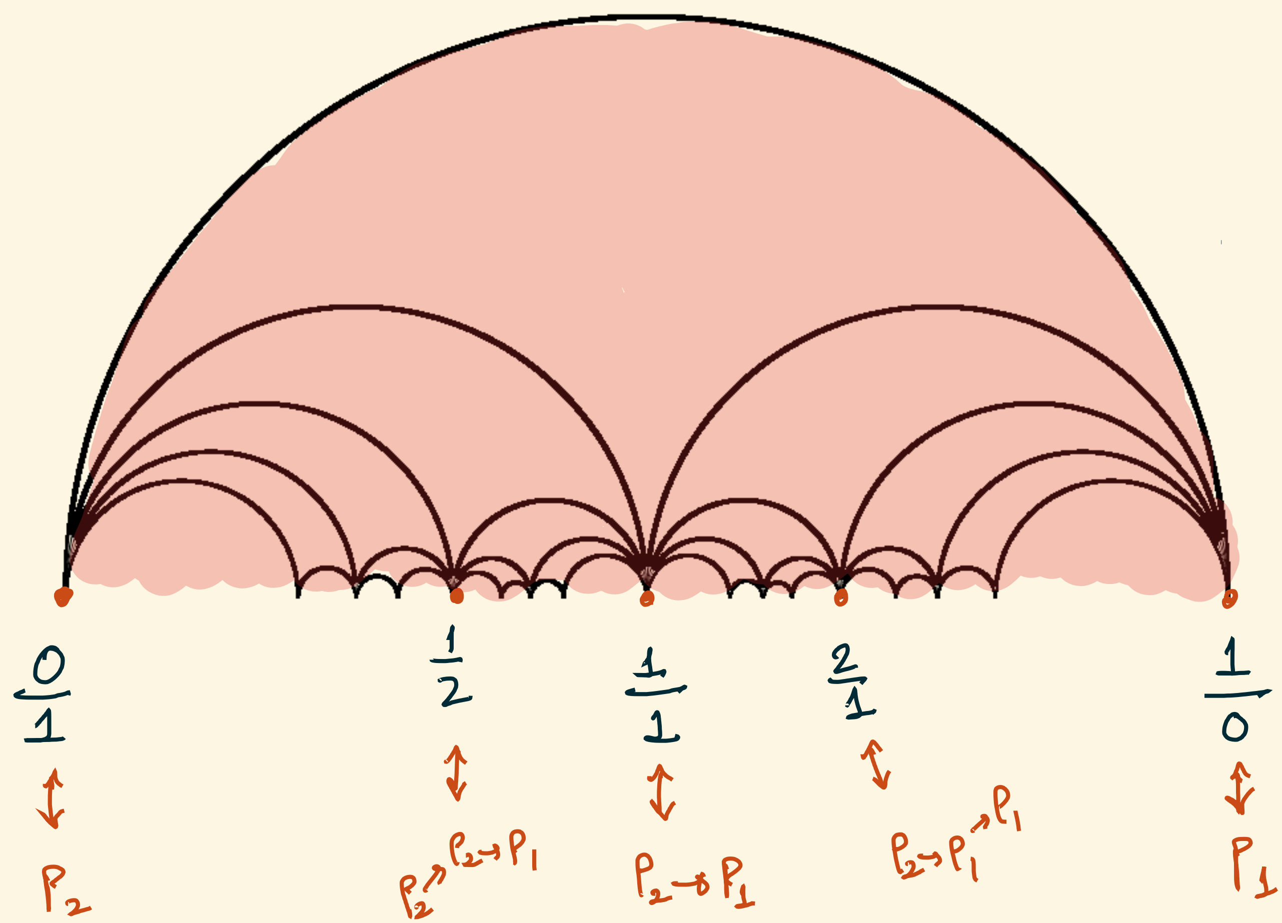
Theorem [BDL]: For $q = 1$,

① hom and occ coincide.

② $\hom_X \mapsto \pm \dfrac{\hom(X, P_2)}{\hom(X, P_1)}$ gives

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{Sphericals} \\ \text{of } \mathcal{C} \end{array} \right\} & \xleftrightarrow{1:1} & \mathbb{Q} \cup \{\infty\} \\ \circlearrowleft & & \circlearrowleft \\ B_3 & & B_3 \end{array}$$

The story at $q = 1$



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The q-deformed story

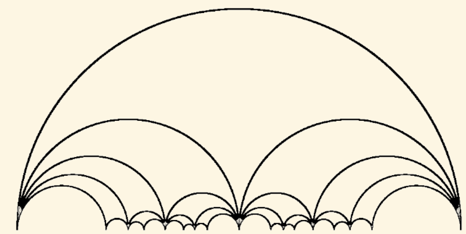
Question : Can we recover q-rationals via some deformation of the quotients $\pm \dfrac{\hom(X, P_2)}{\hom(X, P_1)}$?

Answer : Yes, and more!

Fractional linear transformations, again

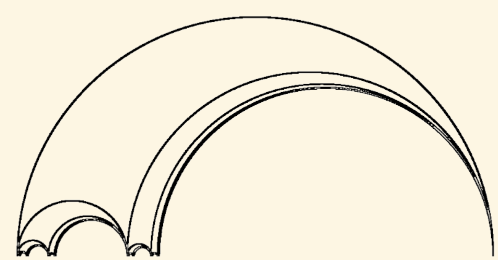
$$B_3 \longrightarrow PSL_2(\mathbb{Z})$$

$$\sigma_1 \longmapsto \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \longmapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



$$B_3 \longrightarrow PSL_2(\mathbb{Z}[q^{\pm}])$$

$$\sigma_1 \longmapsto \begin{bmatrix} q^{-1} & -q^{-1} \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 \longmapsto \begin{bmatrix} 1 & 0 \\ 1 & q^{-1} \end{bmatrix}$$



Left & right q-rationals

$$B_3 \longrightarrow PSL_2(\mathbb{Z}[q^{\pm}])$$

Right q-rationals: B_3- orbit of $\infty = \frac{1}{0}$

(Denoted $\left[\frac{r}{s}\right]^{\#}$)

Left q-rationals: B_3- orbit of $\frac{1}{1-q}$.

(Denoted $\left[\frac{r}{s}\right]^{\flat}$)

The q-deformed story for B_3

Theorem [B.-Becker-Licata]

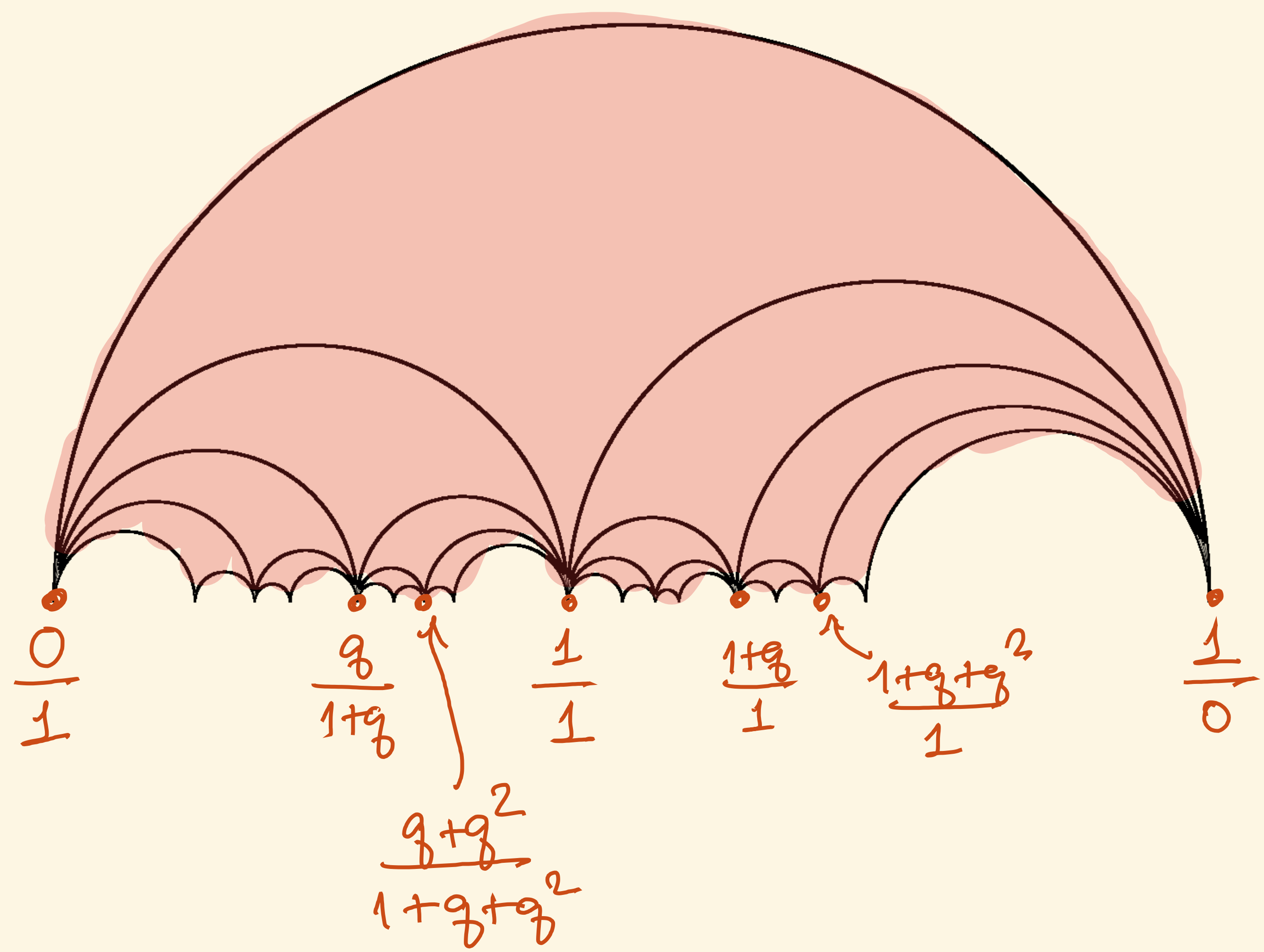
① $\pm q^{(\,)} \dfrac{\mathrm{occ}(P_2, X)}{\mathrm{occ}(P_1, X)}$ are the right q-rationals.

② $\pm q^{(\,)} \dfrac{\overline{\hom}(X, P_2)}{\overline{\hom}(X, P_1)}$ are our left q-rationals.

③ Everything is B_3-equivariant.

The q-deformed story for B_3

The right q-rationals at $q \neq 1$:



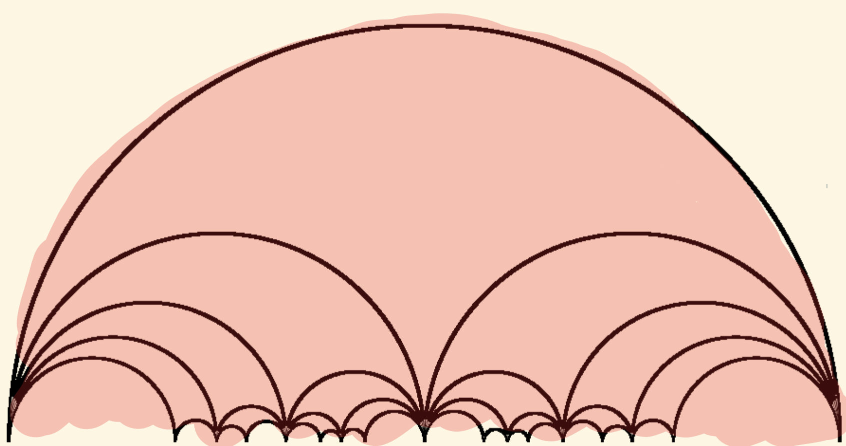
Specialising q

Set $q = 1$.

Consider the ideal triangle with vertices $0, 1, \infty$.

[corresponds to a piece of Stab $\mathcal{C}$]

The $PSL_2(\mathbb{Z})$-orbit:



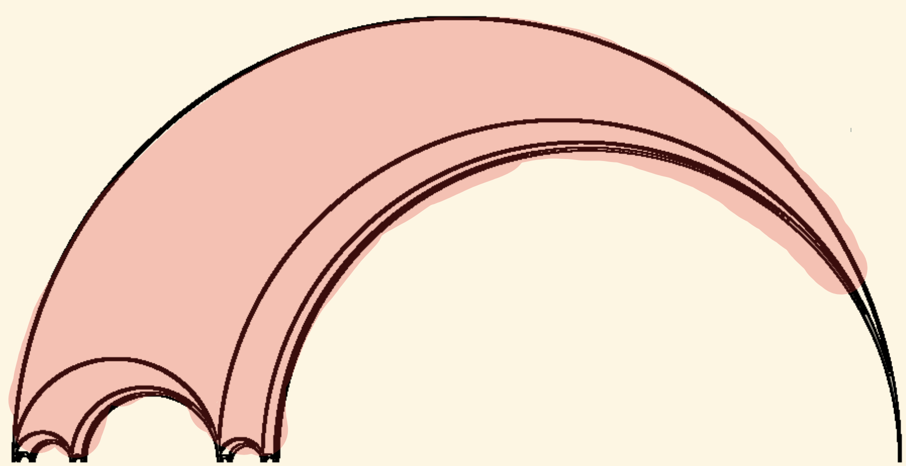
[$q = 1$]

Specialising q

Now fix $0 < q < 1$.

Consider the ideal triangle with vertices $0, 1, \infty$.

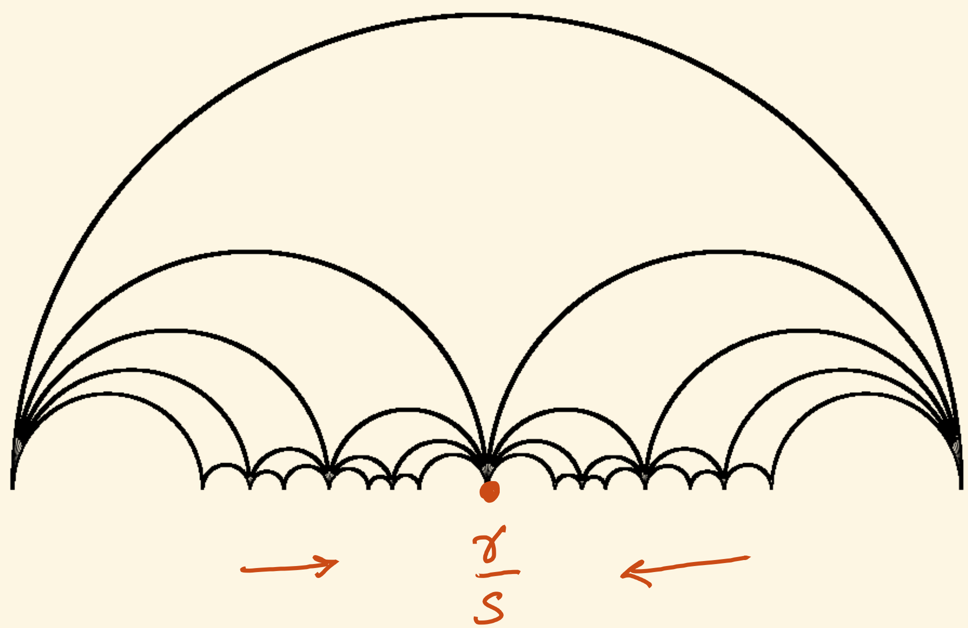
The $PSL_{2,q}(\mathbb{Z})$-orbit:



$[q = 0.3]$

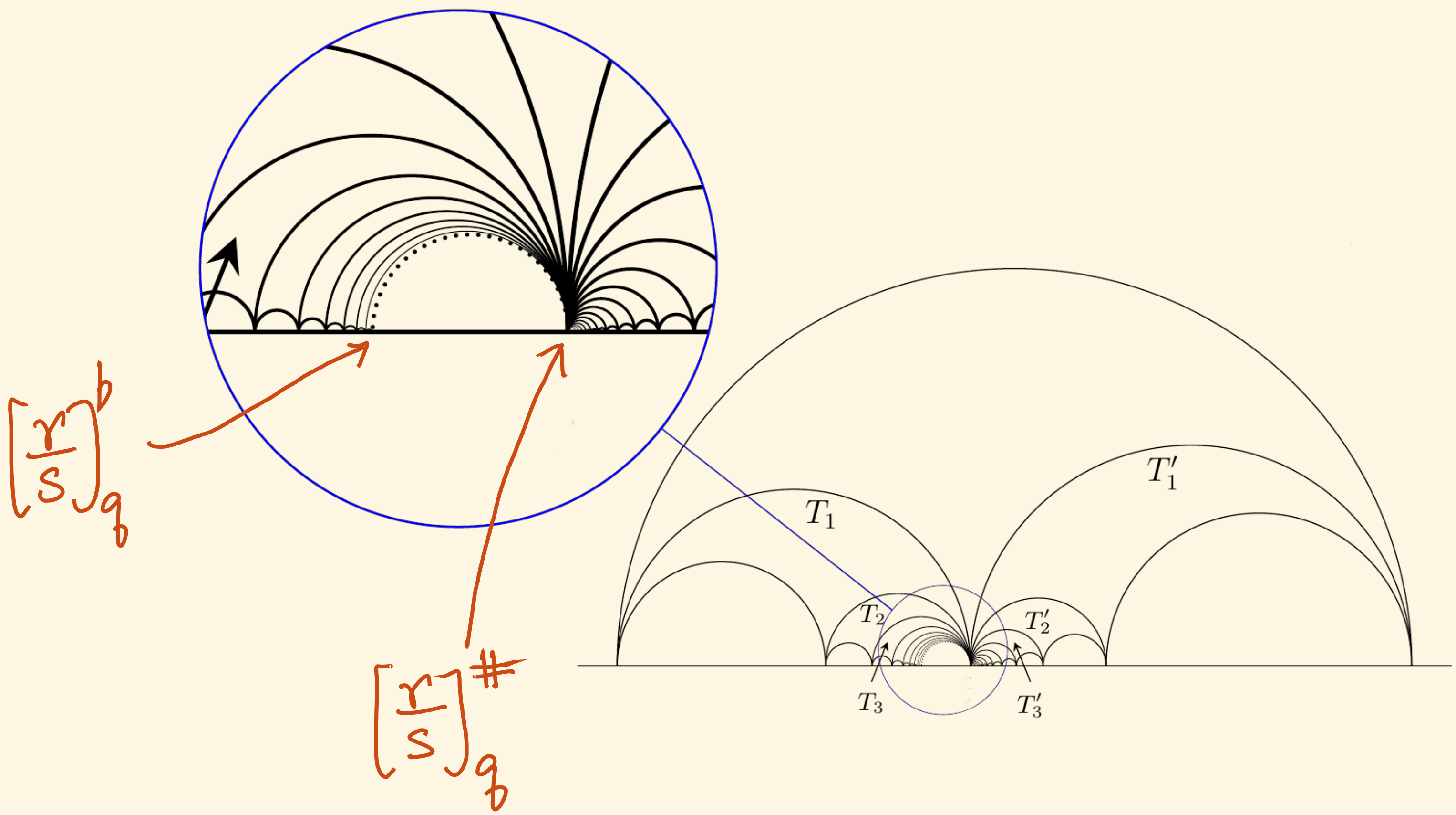
Specialising q

At $q=1$, left & right limits of Farey triangles agree.



Specialising q

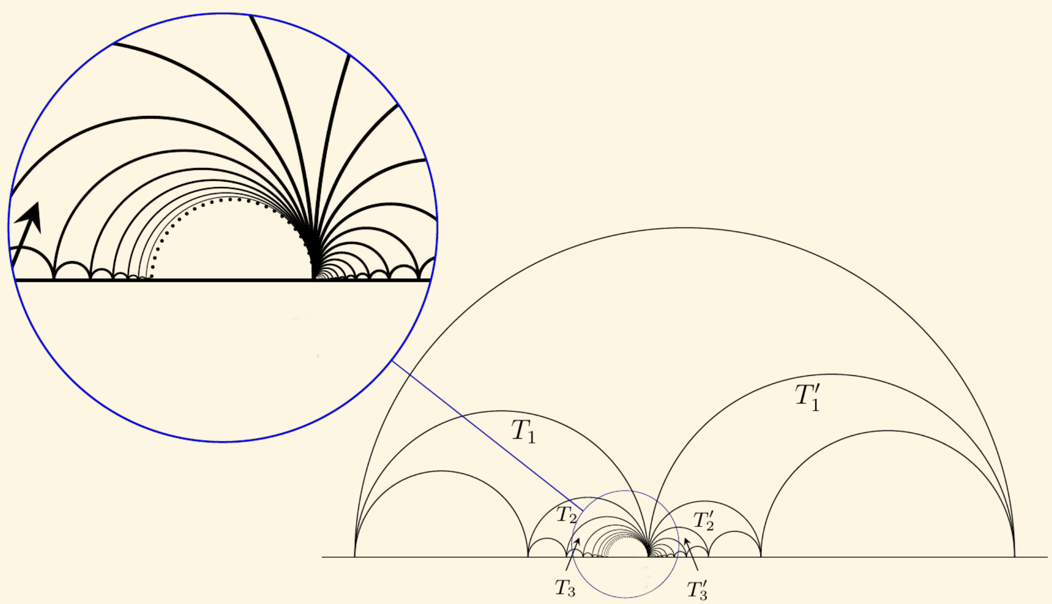
At $q \neq 1$, the left & right limits of Farey triangles do not agree – we get $\left[\frac{r}{s}\right]_q^\flat$ & $\left[\frac{r}{s}\right]_q^\sharp$!



Specialising q

At $q \neq 1$, the left & right limits of Farey triangles do not agree — we get $\left[\frac{r}{s}\right]^{\flat}_q$ & $\left[\frac{r}{s}\right]^{\sharp}_q$!

Moreover, the entire semicircle connecting them lies in the limit.



$\overline{\mathrm{Stab}}^q \mathcal{C}$ at a fixed positive q

Thm [B-Becker-Licata]

1. The union of the closed semicircles $\left[\left[\frac{r}{s}\right]_q^\flat , \left[\frac{r}{s}\right]_q^\sharp \right]$ is dense in the boundary of $\overline{\mathrm{Stab}}^q \mathcal{C}$

2. The remaining points of the boundary are exactly the "q-irrationals".

3. The boundary is homeomorphic to S^1.

Selected subsequent work

- Jouteur, Morier-Genoud, Leclere, Ovsienko, Veselov, ... ; q-rationals & irrationals

- Ren et al : homological interpretation

- Fan, Qiu : topological interpretation

- Thomas : connection to differential operators

Thank you!