A Thurston Compactification of the Space of Stability Conditions

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Joint work with
Anand Deopurkar and
Anthony Licata

I. Overview

I. OVERVIEW

Σ hyperbolic surface

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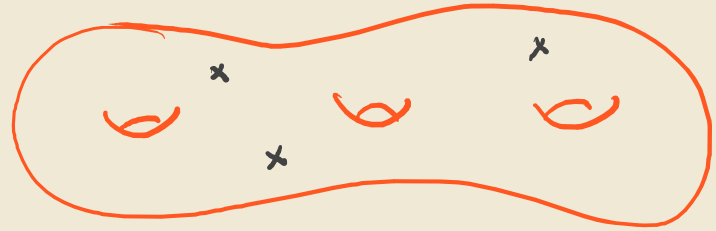
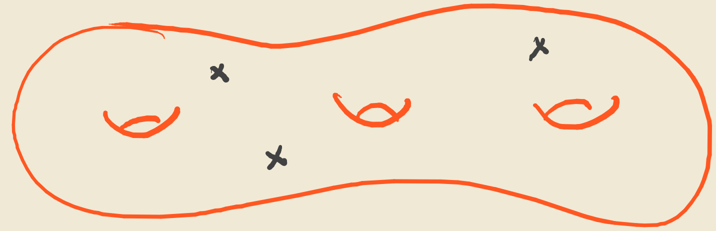


$\rightsquigarrow$

Teich(Σ) $\circlearrowleft$ Mapping class group(Σ)

I. OVERVIEW

Σ hyperbolic surface



$$\rightsquigarrow$$

$\mathrm{Teich}(\Sigma) \circlearrowleft$ Mapping class group (Σ)

[Thurston] $\downarrow$

$\overline{\mathrm{Teich}(\Sigma)} \circlearrowleft \mathrm{MCG}(\Sigma)$

I. OVERVIEW

$\mathcal{T}$ triangulated category

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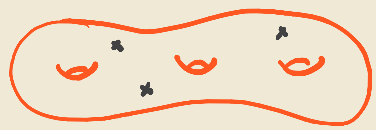
$$\begin{tikzcd}
\mathcal{T} \text{ triangulated category} \arrow[d, squiggly] \\
\mathrm{Stab}(\mathcal{T}) \circlearrowleft \mathrm{Aut}(\mathcal{T}) \arrow[d] \\
?
\end{tikzcd}$$

Bridgeland stability space

I. OVERVIEW

$\Sigma =$  $\longleftrightarrow \quad T$

$$
\begin{tikzcd}
\mathrm{Teich} \circlearrowleft \mathrm{MCG} \arrow[d, hook] \arrow[r, leftrightarrow] & \mathrm{Stab} \circlearrowleft \mathrm{Aut}(T) \\
\overline{\mathrm{Teich}} \circlearrowleft \mathrm{MCG} &
\end{tikzcd}
$$

I. OVERVIEW

$\Sigma =$  $\longleftrightarrow \quad T$

$$
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\end{tikzcd}
$$

GOALS FOR TODAY:

- A general (conjectural) recipe for $\overline{\mathrm{Stab}}$, following Thurston
- Theorems for the A_2 case

II. Setup

II SETUP

Consider triangulated category $\mathcal{T}$.

Comes with

- Shift equivalence $[1]$
- Distinguished triangles

$$A \to B \to C \to A[1]$$

II SETUP

Consider triangulated category $\mathcal{T}$.

Often, we have bounded t-structures

<u>Example</u>

$$\mathrm{Rep}(A) \subseteq D^b\mathrm{Rep}(A)$$

$$\cdots \;\Big|\; \mathrm{Rep}(A)[1] \;\Big|\; \mathrm{Rep}(A) \;\Big|\; \mathrm{Rep}(A)[-1] \;\Big|\; \cdots$$

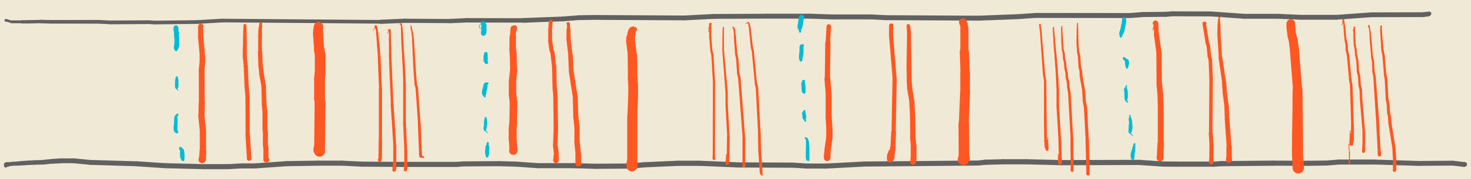
II SETUP

Consider triangulated category $\mathcal{T}$.

Often, we have bounded t-structures

Sometimes, we also have

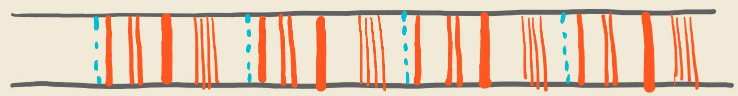
Bridgeland stability conditions



III. Stability Conditions

III STABILITY CONDITIONS

A stability condition is specified by:

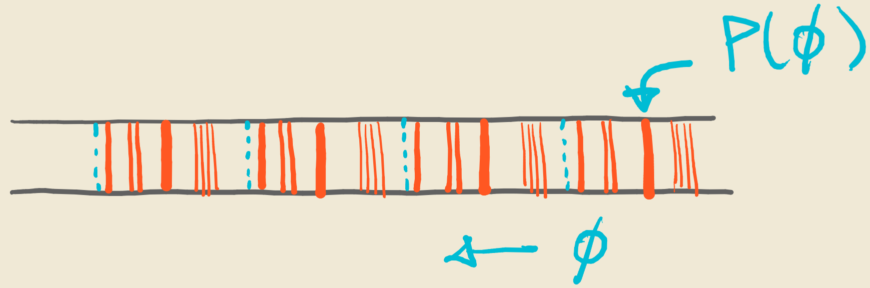
① A "slicing" $\mathcal{P}$ 

into additive subcategories ($\mathbb{R}$-indexed)

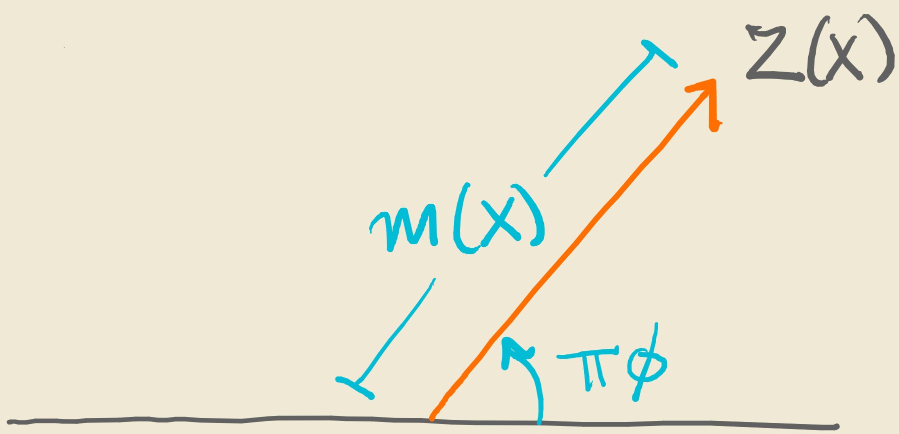
② $Z : K(\mathcal{T}) \to \mathbb{C}$ group homomorphism

+ compatibility conditions

III STABILITY CONDITIONS



- Objects in any $P(\phi)$ are <u>semistable</u>.
- If $X \in P(\phi)$, then $Z(X)$ has <u>phase</u> ϕ and <u>mass</u> $|Z(X)|$



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- mass $:= \sum$ masses of ss factors

III STABILITY CONDITIONS

Theorem (Bridgeland)

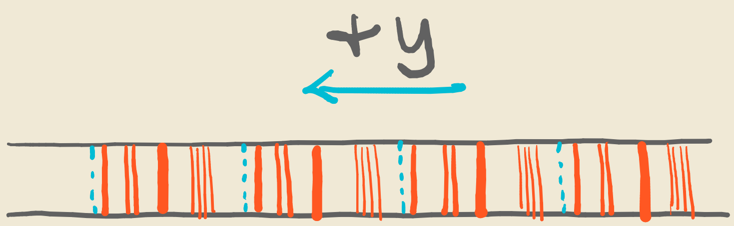
The collection of all stability conditions on $\mathcal{T}$ is a complex manifold

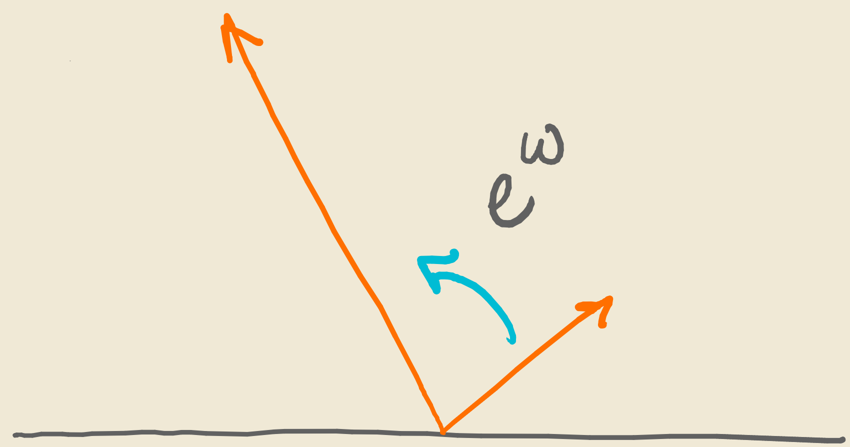
III STABILITY CONDITIONS

$\mathbb{C}$ acts on stability conditions:

For $\omega = x + i\pi y$,

$$\omega \cdot (P, Z) = (P + y,\ e^{\omega} Z)$$





III STABILITY CONDITIONS

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<u>Definition</u>

$$\mathrm{Stab}(T) := \{\text{stab conditions on } T\} / \mathbb{C}$$

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Definition

$$\mathrm{Stab}(T) := \{\text{stab conditions on } T\}/\mathbb{C}$$

Remark

$$\mathrm{Aut}(T) \circlearrowright \mathrm{Stab}(T).$$

IV Recipe for Compactification (following Thurston)

IV RECIPE FOR COMPACTIFICATION

Fix $\mathcal{S}$, a nice subset of objects of $\mathcal{T}$.

Examples

- All possible stable objects
- All "spherical" objects

IV RECIPE FOR COMPACTIFICATION

Fix S, a nice subset of objects of T.

We have

$$\mathrm{Stab}(T) \longrightarrow \mathbb{P}(\mathbb{R}^S)$$

IV RECIPE FOR COMPACTIFICATION

Fix S, a nice subset of objects of T.

We have

$$\begin{aligned} \mathrm{Stab}(T) &\xrightarrow{m} \mathbb{P}(\mathbb{R}^S) \\ \tau &\longmapsto [x \mapsto m_\tau(x)] \end{aligned}$$

mass

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Fix S, a nice subset of objects of T.

We have

$$\begin{aligned} \mathrm{Stab}(T) &\xrightarrow{\quad m \quad} \mathbb{P}(\mathbb{R}^S) \\ \tau &\longmapsto [x \mapsto m_\tau(x)] \end{aligned}$$

mass

Plan

Consider closure of $m(\mathrm{Stab}(T))$.

IV RECIPE FOR COMPACTIFICATION

Fix $\mathcal{S}$, a nice subset of objects of T.

We also have

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathbb{P}(\mathbb{R}^{\mathcal{S}}) \\ Y & \longmapsto & [X \mapsto \dim \mathrm{Hom}_T(X,Y)] \end{array}$$

IV RECIPE FOR COMPACTIFICATION

$$
\begin{tikzcd}
S \arrow[dr, hook] & \\
& \mathbb{P}(\mathbb{R}^S) \\
\mathrm{Stab}(T) \arrow[ur, hook] &
\end{tikzcd}
$$

IV RECIPE FOR COMPACTIFICATION

Elements of S

are limits of sequences in $\mathrm{Stab}(T)$!

$$\begin{tikzcd} S \arrow[dr, hook] & \\ & \mathbb{P}(\mathbb{R}^S) \\ \mathrm{Stab}(T) \arrow[ur, hook] & \end{tikzcd}$$

V Conjectures

V CONJECTURES

- $\mathrm{Stab}(\mathcal{T}) \hookrightarrow \mathbb{P}(\mathbb{R}^S)$ is injective and a homeomorphism onto its image.
- Its closure is a compact real manifold with boundary.
- $S \subseteq$ boundary, and is dense.

VI The A_2 Case

VI THE A_2 CASE

$\mathcal{T}$ generated by P_1 & P_2 :

- $\mathrm{Hom}^n(P_i, P_i) = \begin{cases} \mathbb{C} & n = 0, 2 \\ 0 & \text{otherwise} \end{cases}$ (spherical)

- $\mathrm{Hom}^n(P_i, P_j) = \begin{cases} \mathbb{C} & n = 1 \\ 0 & \text{otherwise} \end{cases}$

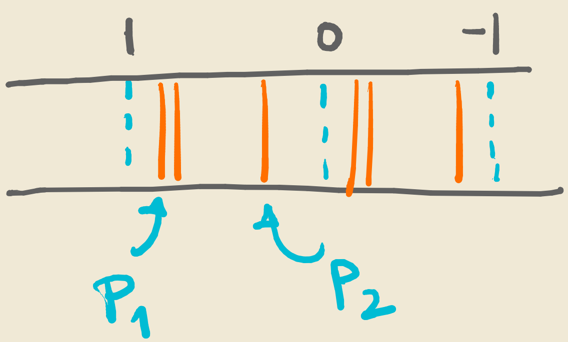
- 2-Calabi-Yau

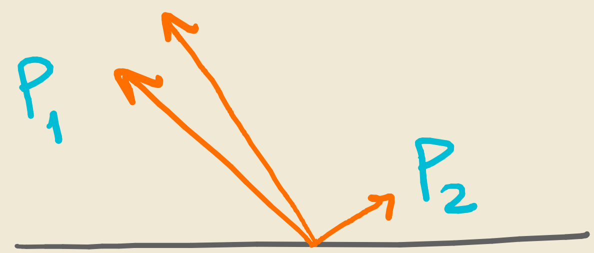
$$\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, A[2])^*$$

VI THE A_2 CASE

$\mathcal{T}$ generated by P_1 & P_2.

Easy to write down stability conditions:





VI THE A_2 CASE

Aut(T) contains spherical twists

σ_{P_1} & σ_{P_2}

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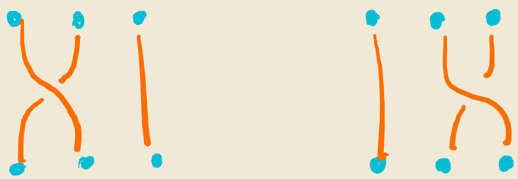
$\mathrm{Aut}(\mathcal{T})$ contains spherical twists

σ_{P_1} & σ_{P_2} $\rightsquigarrow$ generate B_3!



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$$B_3/\text{centre} \simeq PSL_2(\mathbb{Z})$$

VI THE A_2 CASE

$\mathrm{Aut}(\mathcal{T})$ contains spherical twists

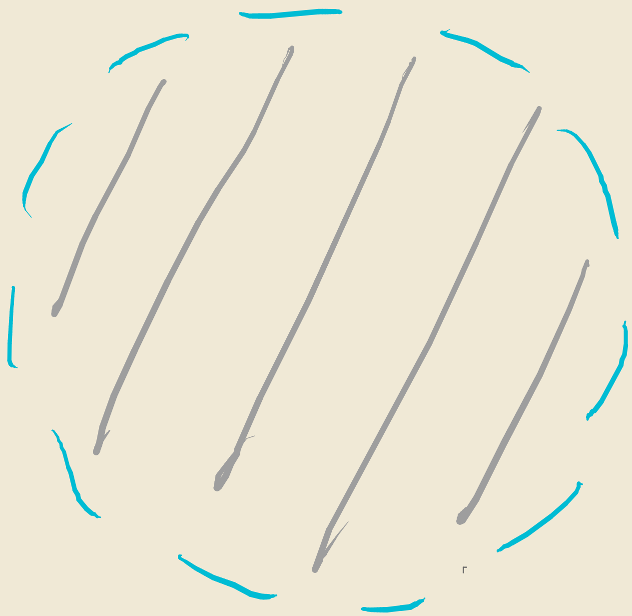
σ_{P_1} & σ_{P_2} $\rightsquigarrow$ generate B_3!

$B_3/\text{centre} \simeq PSL_2(\mathbb{Z})$

$B_3 \circlearrowright \mathcal{T}$ gives $PSL_2(\mathbb{Z}) \circlearrowright \mathrm{Stab}(\mathcal{T})$

VI THE A_2 CASE

$PSL_2(\mathbb{Z})$ acts on hyperbolic plane,



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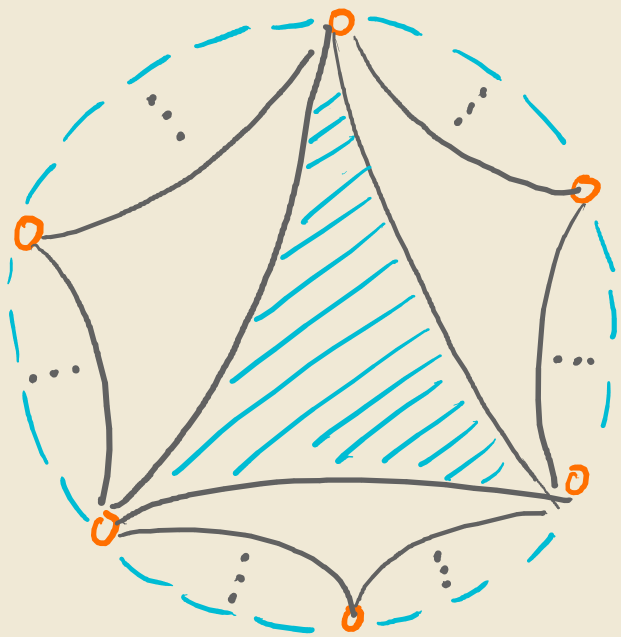
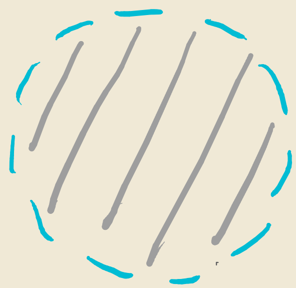


tessellated by ideal triangles that are permuted by $PSL_2(\mathbb{Z})$

VI THE A_2 CASE

Theorem (Thomas, Bridgeland-Qiu-Sutherland, Ikeda)

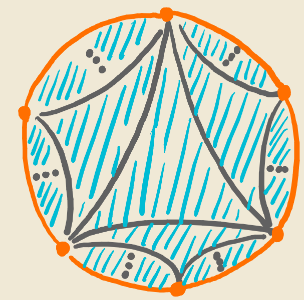
$\exists$ $PSL_2(\mathbb{Z})$-equivariant homeomorphism

$\mathrm{Stab}(T) \xrightarrow{\sim}$ 

VI THE A_2 CASE

<u>Theorem</u> (–)

- $\overline{\mathrm{Stab}}(T) \simeq$ closed unit ball



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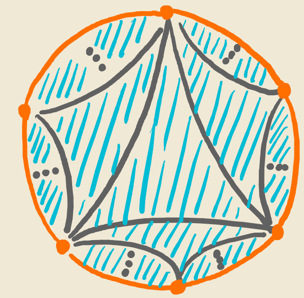
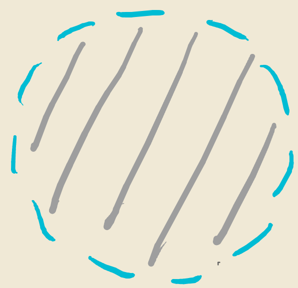
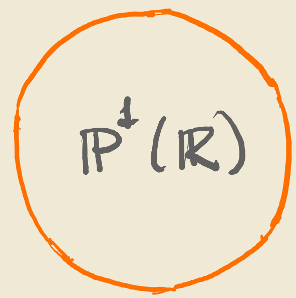


$=$  $\amalg$ 

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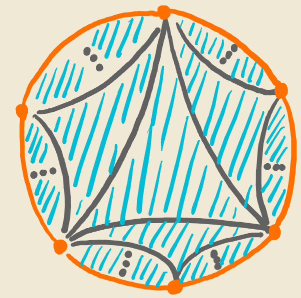
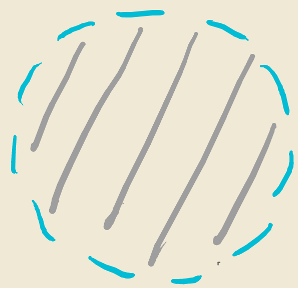


$=$  $\sqcup$ $\mathbb{P}^1(\mathbb{R})$

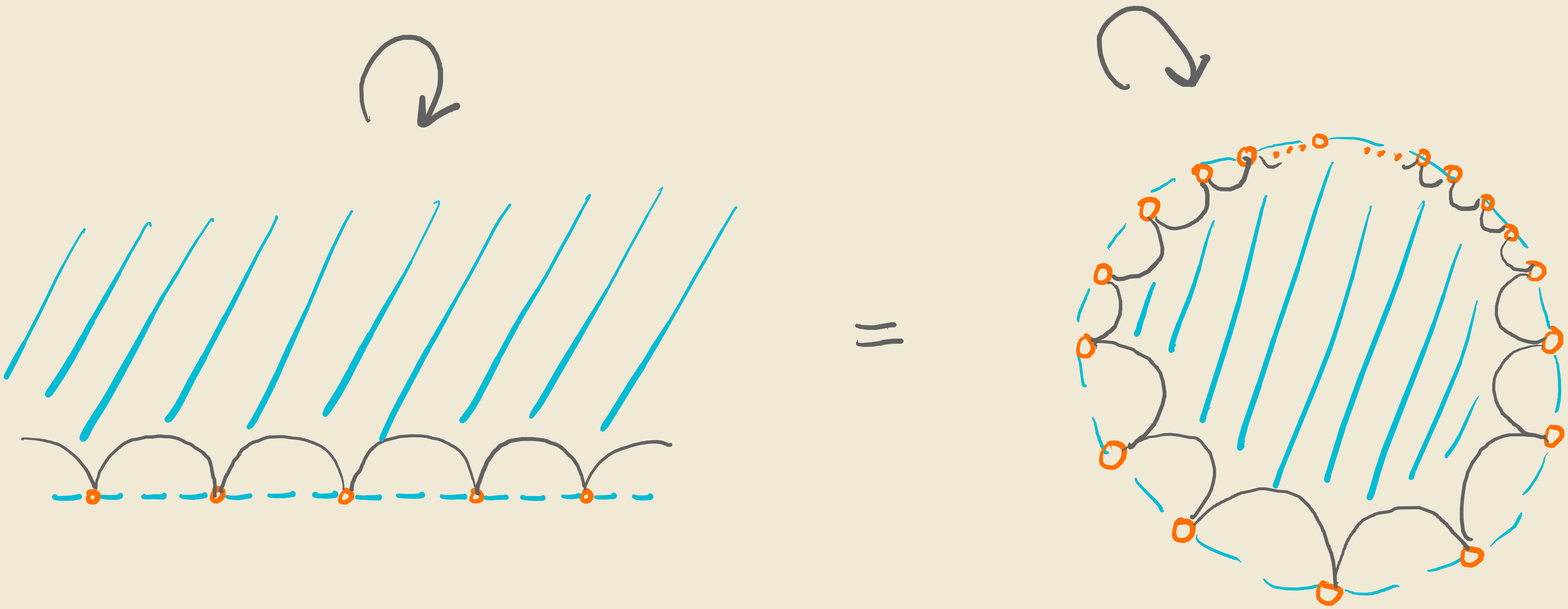
- $S \longleftrightarrow$ vertices of ideal triangles

$\simeq$ $\mathbb{P}^1(\mathbb{Q}) \subseteq \mathbb{P}^1(\mathbb{R})$

THANKS !

EXAMPLE IN PROGRESS : $\hat{A}_1$

$$\langle \sigma_1^2, \sigma_2^2 \rangle \leq PSL_2(\mathbb{Z})$$



VI THE A_2 CASE

Why is $S \subseteq \overline{\mathrm{Stab}}(\mathcal{T})$?

Let $A \in S$. σ_A = spherical twist in A

Recall $m : \tau \mapsto [X \mapsto m_\tau(X)]$

$$m_{\sigma_A^n \tau}(X) = m_\tau(\sigma_A^{-n} X)$$

Projectively, looks like $\dim \mathrm{Hom}(A, X)$!