MATH 2301 30 Jul 2020

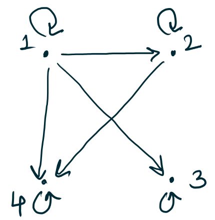
* Recap: Discussed sets, relations, functions. We'll revisit these topics periodically.

* Today's topic: more foundations.

* Graphs: A great way to organise info visually, especially info about relations.

Defn: A directed graph consists of a vertex set V and an edge set $\underbrace{E \subseteq V \times V}_{\text{just a relation on } V!}$

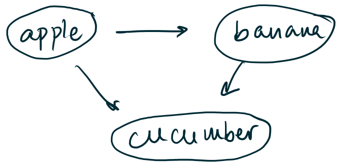
E.g. $V = \{1,2,3,4\}$, $E = \{(a,b) \mid \underbrace{a|b}_{b \text{ is an integer multiple of } a}\}$



You can draw a picture of any relation this way!
If $R \subseteq S \times S$, make S your vertex set, and R is the edge set.

E.g. $V = \{\text{apple, banana, cucumber}\}$

$E = \{(x,y) \mid x - y \text{ is in alphabetical order}\}$



* If $(a,b) \in E$ in a (directed) graph (V,E) then we'll write $a \to b$

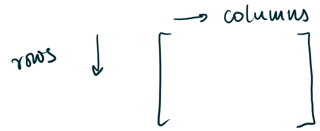
Defn: An undirected graph is the following: Again, there is a vertex set V & an edge set $E \subseteq V \times V$. Moreover, E has the special property that if $(a,b) \in E$ then $(b,a) \in E$.

In this case we think of the pair of edges as being a single undirected edge $a - b$.

[Sometimes, we can think of this pair (a,b) & (b,a) together as an unordered pair $\{a,b\}$.]

* Adjacency matrix

Defn: A matrix is a rectangular array of (usually) numbers. An $m \times n$ matrix M has m rows & n columns



The entry in the i^{th} row & j^{th} column is denoted M_{ij}

E.g. If $M = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$, then $M_{13} = 4$, $M_{22} = 2$

Let (V,E) be a graph. To construct an adjacency matrix, we first order our vertices: $V = (v_1, v_2, \cdots, v_n)$

Then construct a matrix A with n rows & n columns.

Put a 1 in the $(i,j)^{th}$ spot if there is $v_i \to v_j$

Otherwise put a 0 in $(i,j)^{th}$ spot.

i.e. $A_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{if } (v_i, v_j) \notin E. \end{cases}$

Rmk: Choosing a different order on V will give you a different-looking adjacency matrix.

Notes:

* A encodes the entire structure of the graph (V,E)
* Extremely useful for calculations.
* Depends on the ordering on V.

* Some desirable properties of relations: on a single set.

① Reflexivity: Let $R \subseteq S \times S$. Then R is reflexive if $(s,s) \in R$ for each $s \in S$.

② Symmetry: Let $R \subseteq S \times S$. Then R is symmetric if whenever $(a,b) \in R$, we also have $(b,a) \in R$. for every $a,b \in S$.

③ Anti-symmetry: Let $R \subseteq S \times S$. Then R is anti-symmetric if whenever $(a,b) \in R$ and $(b,a) \in R$, we must have $a=b$.

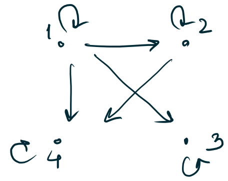
④ Transitivity: Let $R \subseteq S \times S$. Then R is transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, we also have $(a,c) \in R$.

Examples:

(1) Let $R = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a+b \text{ is odd}\}$.
- Not reflexive
- Is symmetric
- Not anti-symmetric
- Not transitive: e.g. $(1,2) \in R$ and $(2,5) \in R$ but $(1,5) \notin R$.

(2) See previous examples and figure out their properties.

E.g. (adjacency matrix)



$\xrightarrow[\text{matrix}]{\text{adj}}$

$$
\begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array}
\begin{array}{c}
\begin{array}{cccc} (1) & (2) & (3) & (4) \end{array} \\
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

Notes: * A relation is reflexive iff all diagonal entries are 1.
if and only if

* Similarly, R is symmetric iff $A_{ij} = A_{ji}$ ($A_{ij} = 1$ iff $A_{ji} = 1$), ie the matrix is symmetric. i.e equal if you flip across the diagonal.

* R is anti-symmetric iff whenever $i \neq j$ and $A_{ij} = 1$, then $A_{ji} = 0$.

* R is transitive iff ??? [we'll get back to this]

Closures of relations: Next time.