MATH2301 31 July 2020

* Recap: Discussed graphs, adjacency matrices, and several properties of relations.

* Today: More about properties, equivalence relations

* Clarification about symmetry vs anti-symmetry. (properties of $R \subseteq S \times S$)

symmetry: if $(a,b) \in R$ then $(b,a) \in R$

anti-symmetry: if $(a,b) \in R$ and $a \neq b$ then $(b,a) \notin R$.

They are close to being "opposite" to each other, but not quite.

Suppose R is both symmetric & anti-symmetric.

Then the only possible elements of R are of the form (a,a)

Closures of relations:

Let's focus on reflexivity, symmetry, and transitivity.

Look at $S \times S \subseteq S \times S$. This is a relation on S!

* Trivially satisfies reflexivity, symmetry, transitivity.

If $R \subseteq S \times S$ some other relation. We'd like to ask about the "reflexive closure" of R.

Informally, want this to be the smallest reflexive relation that contains R.

Defn: The reflexive closure of R is a relation R' such that:

similar for symm/transitive closures

① R' is reflexive

② $R \subseteq R'$

③ If $T \subseteq S \times S$ such that $R \subseteq T \subsetneq R'$ (strictly), then T cannot be reflexive.

* Note: If R is already reflexive, then its reflexive closure is just R.

Q: How to construct the reflexive closure?

A: Set $R' = R \cup \{(x,x) \mid x \in S\}$.

Q: Symmetric closure?

A: Set $R' = R \cup \{(b,a) \mid (a,b) \in R\}$

Q Transitive closure?

A: More complicated ... we'll revisit.

In terms of adjacency matrices:

$$\begin{array}{c|c} & v_1 \cdots v_n \\ \hline \begin{matrix} v_1 \\ \vdots \\ v_n \end{matrix} & \begin{bmatrix} & 1 \\ 0 & \end{bmatrix} \end{array}$$

$\begin{bmatrix} \ddots \end{bmatrix}$ terms of the form (x,x) ⇝ reflexive closure changes each diagonal entry to 1.

(i,j), (j,i) ⇝ symmetric closure by "symmetrising" matrix: if $A_{ij} = 1$, make $A_{ji} = 1$

Transitive closure??

Equivalence relations

Defn: A relation $R \subseteq S \times S$ is called an equivalence relation if it is reflexive, symmetric, and transitive.

* In this case, we often write $x \sim_R y$ or $x \sim y$ to mean that $(x,y) \in R$.

* This is a powerful combination: if two objects are related, we can sort of treat them as being equivalent.

Example: $\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x - y \text{ is even}\} = R$

- $(x,x) \in R$ b/c $x - x = 0$ is even (refl)
- If $x-y$ is even then $y-x$ is even (sym)
- If $x-y$ is even they have the same parity, similarly if $y-z$ even, they have the same parity

So x & z have the same parity (transitive)

Equivalence classes:

Let $R \subseteq S \times S$ be an equivalence relation.

Let $x \in S$. We say that the set

$[x] = \{ y \in S \mid x \sim y \}$ is called the

equivalence class of x.

Example: $\{ (x, y) \mid x - y \text{ even} \}$ on $\mathbb{Z} \times \mathbb{Z}$

$[2] = \{ y \in S \mid 2 - y \text{ is even} \}$

$= \{ \cdots, -2, 0, 2, 4, 6, \cdots \}$

$[3] = \{ \cdots, -1, 1, 3, 5, \cdots \}$

$[4] = \{ y \in S \mid 4 - y \text{ is even} \} = [2]$

In this example, :

$\cdots [-4] = [-2] = [0] = [2] = [4] = \cdots$ ↙ evens

$\cdots [-3] = [-1] = [1] = [3] \cdots$ ↖ odds.

Prop: If $R \subseteq S \times S$ is an equivalence relation, then R partitions S into disjoint subsets, whose union is S.

Proof:

① If $x \in S$ then $x \in [x]$ (reflexivity)

② If $\underbrace{y \in [x] \text{ then } x \in [y]}_{\text{true because of symmetry.}}$ and in fact $[x] = [y]$.

First suppose that $y \in [x]$. We'll show $[x] = [y]$

Suppose $z \in [x] \Rightarrow \boxed{x \sim z}$

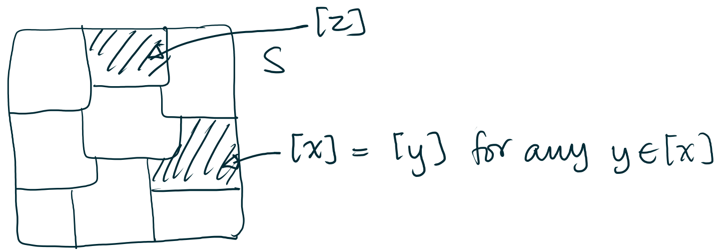
We know $y \in [x]$, so $x \sim y$; by symmetry $\boxed{y \sim x}$

By transitivity, see that $y \sim z \Rightarrow z \in [y]$

Similarly, show that if $z \in [y]$, then $z \in [x]$.

Together, this means $[x] = [y]$.

Pictorially:



If x & z live in different buckets, they are not related. All elts in a single bucket are related to each other.

Example: $\mathbb{Z} = \{\text{evens}\} \amalg \{\text{odds}\}$
↑ disjoint union

Application: Modular arithmetic [Next week]