

* Recap: Discussed graphs, adjacency matrices, and several properties of relations.

* Today: More about properties, equivalence relations

* Clarification about symmetry vs anti-symmetry. (properties of $R \subseteq S \times S$)

symmetry: if $(a,b) \in R$ then $(b,a) \in R$

anti-symmetry: if $(a,b) \in R$ and $a \neq b$ then $(b,a) \notin R$.

They are close to being "opposite" to each other, but not quite. Suppose R is both symmetric & anti-symmetric.

Then the only possible elements of R are of the form (a,a)

Closures of relations:

Let's focus on reflexivity, symmetry, and transitivity.

Look at $S \times S \subseteq S \times S$. This is a relation on S !

* Trivially satisfies reflexivity, symmetry, transitivity.

If $R \subseteq S \times S$ some other relation. We'd like to ask about the "reflexive closure" of R .

Informally, want this to be the smallest reflexive relation that contains R .

Defn: The reflexive closure of R is a relation R' such that:

① R' is reflexive

② $R \subseteq R'$

③ If $T \subseteq S \times S$ such that $R \subseteq T \subsetneq R'$, then T cannot be reflexive

similar for symm/transitive closures

* Note: If R is already reflexive, then its reflexive closure is just R .

Q: How to construct the reflexive closure?

A: Set $R' = R \cup \{(x, x) \mid x \in S\}$.

Q: Symmetric closure?


A: Set $R' = R \cup \{(b, a) \mid (a, b) \in R\}$

Q: Transitive closure?

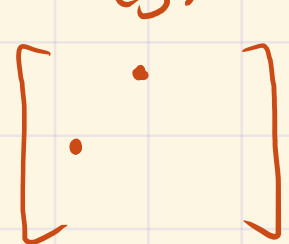
A: More complicated -- we'll revisit.

$$\begin{matrix} v_1 & \dots & v_n \\ \vdots & & \vdots \\ v_n & & 0 \end{matrix} \begin{bmatrix} & & & \\ & & & \\ & & 1 & \\ & & & \end{bmatrix}$$

In terms of adjacency matrices:

 terms of the form (x, x)

\leadsto reflexive closure changes each diagonal entry to 1.

(i, j) 

\leadsto symmetric closure by "symmetrising" matrix:
if $A_{ij} = 1$, make $A_{ji} = 1$

Transitive closure??

Equivalence relations

Defn: A relation $R \subseteq S \times S$ is called an equivalence relation if it is reflexive, symmetric, and transitive.

* In this case, we often write $x \sim_R y$ or $x \sim y$ to mean that $(x, y) \in R$.

* This is a powerful combination: if two objects are related, we can sort of treat them as being equivalent.

Example: $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x - y \text{ is even}\} = R$

- $(x, x) \in R$ b/c $x - x = 0$ is even (ref)
- If $x - y$ is even then $y - x$ is even (sym)
- If $x - y$ is even they have the same parity, similarly if $y - z$ even, they have the same parity

So x & z have the same parity (transitive)

Equivalence classes:

Let $R \subseteq S \times S$ be an equivalence relation.

Let $x \in S$. We say that the set

$[x] = \{y \in S \mid x \sim y\}$ is called the equivalence class of x .

Example: $\{(x, y) \mid x - y \text{ even}\}$ on $\mathbb{Z} \times \mathbb{Z}$

$$[2] = \{y \in S \mid 2 - y \text{ is even}\}$$
$$= \{\dots, -2, 0, 2, 4, 6, \dots\}$$

$$[3] = \{\dots, -1, 1, 3, 5, \dots\}$$

$$[4] = \{y \in S \mid 4 - y \text{ is even}\} = [2]$$

In this example, :

$$\dots [-4] = [-2] = [0] = [2] = [4] = \dots \quad \swarrow \text{evens}$$

$$\dots [-3] = [-1] = [1] = [3] \dots$$

$$\begin{array}{cccccc} \textcircled{x} & \textcircled{x} & \textcircled{x} & \textcircled{x} & \textcircled{x} & \textcircled{x} \\ -2 & -1 & 0 & 1 & 2 & \end{array}$$

\swarrow odds.

Prop: If $R \subseteq S \times S$ is an equivalence relation, then R partitions S into disjoint subsets, whose union is S .

Proof:

① If $x \in S$ then $x \in [x]$ (reflexivity)

② If $y \in [x]$ then $x \in [y]$ and in fact $[x] = [y]$.
true because of symmetry.

First suppose that $y \in [x]$. We'll show $[x] = [y]$

Suppose $z \in [x] \Rightarrow x \sim z$

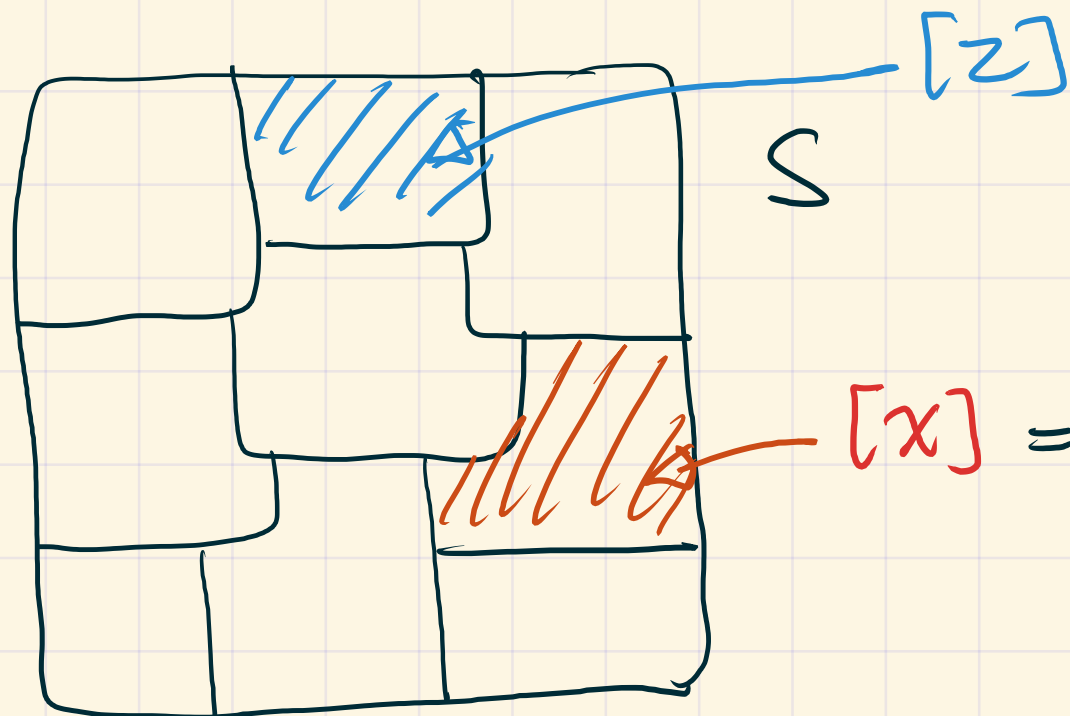
We know $y \in [x]$, so $x \sim y$; by symmetry $y \sim x$

By transitivity, see that $y \sim z \Rightarrow z \in [y]$

Similarly, show that if $z \in [y]$, then $z \in [x]$.

Together, this means $[x] = [y]$.

Pictorially:



If x & z live in different buckets, they are not related. All elts in a single bucket are related to each other.

Example: $\mathbb{Z} = \{\text{evens}\} \sqcup \{\text{odds}\}$

\uparrow disjoint union

Application: Modular arithmetic [Next week]