MATH 2301 05 Aug 2020

* Last time: Equivalence relations & equivalence classes

Example: $R = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} \mid a-b \text{ is even}\}$. [* if $(a,b)\in R$ we say $a \sim b$.]

There are two equivalence classes: {odds} & {evens}

Main result: Let R be an equivalence relation on S.

For any $x \in S$, set $[x] = \{y \in R \mid x \sim y\}$, called the "equivalence class of x".

Then:

* $[x]$ consists of exactly those elements of S that are related to x, and nothing else.
* If $y \in [x]$ then $[x] = [y]$, so equivalence classes can only overlap if they are equal, and
* S is partitioned into a disjoint union of equivalence classes: 

$= S$

Note: $S = S_1 \cup S_2 \cup \cdots \cup S_n$ is a partition if $S_i \cap S_j = \phi$ for $i \neq j$. This is also written as $S = S_1 \amalg S_2 \amalg \cdots \amalg S_n$, or more concisely, $S = \coprod_{i=1}^{n} S_i$

Modular arithmetic

* You have all seen modular arithmetic before, even if you think you haven't!

Baby example: $R = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} \mid a-b \text{ is even}\}$.

$\mathbb{Z} = [0] \amalg [1] = [248] \amalg [55]$

$= [42] \amalg [-11]$

. . .

Since $[248] = [0]$, we say that 0 is a representative of the equiv. class $[0]$, and 248 is also a representative of $[0]$.

$42 + 6 = 48$ (42 ∈ [0], 6 ∈ [0], 48 ∈ [0]) even + even = even

$42 + 9 = 51$ (42 ∈ [0], 9 ∈ [1], 51 ∈ [1]) even + odd = odd. odd + even = odd

$9 + 11 = 20$ (9 ∈ [1], 11 ∈ [1], 20 ∈ [0]) odd + odd = even.

We have defined a new addition: on the set $\{[0],[1]\}$:

$[0] + [0] := [0]$

$[0] + [1] := [1]$

$[1] + [0] := [1]$

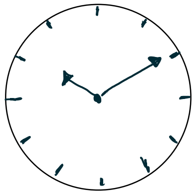
$[1] + [1] := [0]$ ← Loops back to $[0]$!

Q: Why does it make sense? What does that even mean?

Rmk: $\{[0],[1]\} \neq \{0,1\}$

$\{[0],[1]\} = \mathbb{Z}$

* But you know this already:



10 (am) + 9 (hrs) = 7 (pm)

$[10] + [9] = [7]$

5 (pm) + 20 (hrs) = 1 (pm)

$[5] + [20] = [1]$

2 (am) − 4 (hrs) = 10 (pm)

$[2] - [4] = [10]$

(Similar for 24-hour clock)

* We're treating 20 & 8 as equivalent

Fix a positive $d \in \mathbb{N}$, we'll call this the modulus.

Let's construct an equivalence relation

$$R = \sim_d = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid (x-y) \text{ is divisible by } d\}$$

("is divisible by d" $=$ $d \mid (x-y)$)

If $(x,y) \in R$, we say $x \sim_d y$, or more traditionally, we write $x \equiv y \pmod d$

(x is congruent to y modulo d)

Note: If $x \sim_d y$, then there is some integer $n \in \mathbb{Z}$ (positive, negative, or zero) such that

$$x - y = n \cdot d.$$

* Exercise: Check that this is an equivalence relation.

Equivalence classes:

E.g. $[0] = \{0, d, -d, 2d, 5d, -2d, -3d, \dots\}$

$[1] = \{1 + 3d, 1 - 7d, 1, \dots\}$

We (think we) have d equiv. classes:

$[0], [1], \dots, [d-1]$, because $[d] = [0]$

Q: Why are they all different?

(Euclid's division algorithm: if $x \in \mathbb{Z}$, and $d \in \mathbb{N}$, then there is a unique equation

$$x = qd + r, \text{ with } 0 \le r < d.)$$

In fact, they are all different.

Let's define an addition operation:

$$[x] + [y] = [x+y]$$

Q: Why does this make sense?

That is, if $[x] = [a]$, then is it true that

$$[x+y] = [a+y] \, ?$$

In this case, if $[x] = [a]$, then $x - a = nd$.

Then $(x+y) - (a+y) = (x-a) = nd$

$\Rightarrow [x+y] = [a+y]$

Therefore, this addition is well-defined.

E.g. if $d = 3$:

$[1] + [2] = [0]$

$[1] + [1] = [2]$

$[2] + [2] = [4] = [1]$

This is modular addition

Q: Can you multiply modular numbers?

$$[x] \cdot [y] :\overset{?}{=} [xy]$$

[This's a new definition, and we now check that it is sensible]

Does this make sense?

Suppose $[x] = [a]$, i.e. $\boxed{x - a = nd}$ for some $n \in \mathbb{Z}$.

We want $[xy] \stackrel{?}{=} [ay]$

i.e. is $xy - ay$ a multiple of d?

$xy - ay = nyd$ so yes!

$\Rightarrow [xy] = [ay]$, so it is well-defined.

E.g.: If $d = 3$:

$[1] \cdot [1] = [1]$

$[2] \cdot [2] = [1]$