MATH 2301 06 Aug 2020

* Last time: Modular arithmetic (addition & multiplication)

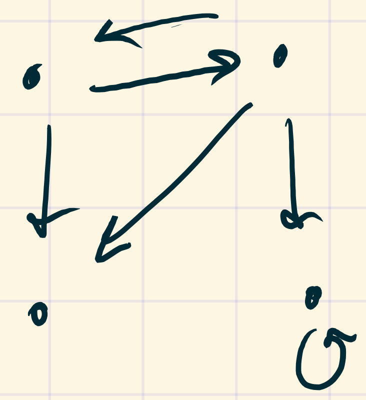
* How many square roots does [1] have modulo 12?

Four: [1], [11], [5], [7]

$[11] = [-1]$, $[7] = [-5]$

* Graphs

Let's review terminology:



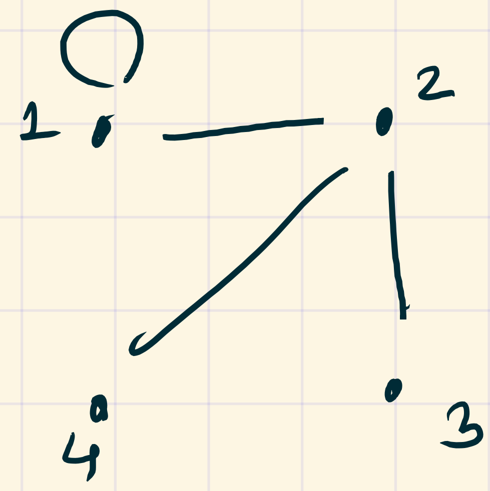
Directed graph or digraph

We have a vertex set V (typically finite)

& edge set $E \subseteq V \times V$

(a,b) means $a \to b$

Undirected graphs:



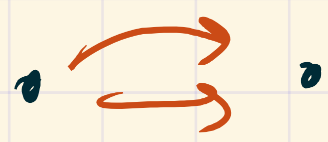
$V = \{1, 2, 3, 4\}$

$E = \{ \boxed{(1,1)}, \boxed{(1,2), (2,1)}, \boxed{(2,4), (4,2)}, \boxed{(2,3), (3,2)} \}$

The edge set is a symmetric relation on V

[To draw, club together $\{(i,j), (j,i)\}$ into a single undirected edge. If $i = j$, then $\{(i,j),(j,i)\} = \{(i,i)\}$]

We mostly consider "simple" graphs:

We disallow multiple edges 

We disallow parallel loops 

Graphs can be used to model all sorts of networks: (network / graph theory)

* Railway / flight network
* Electrical circuits / water pipes
* Facebook friend graph
* World wide web links, ...

(often you want "weighted" graphs, where edges have lengths)

There are some extremely natural questions that you might ask if you look at one of these.

Examples:

1) Is there a route from point A to point B?

2) How long is the route? What's the shortest path?

3) How many routes are there (and how long are they)?

4) How much current/water/-- can flow through the network at full capacity?

(4.5) Can you figure out "clusters"?

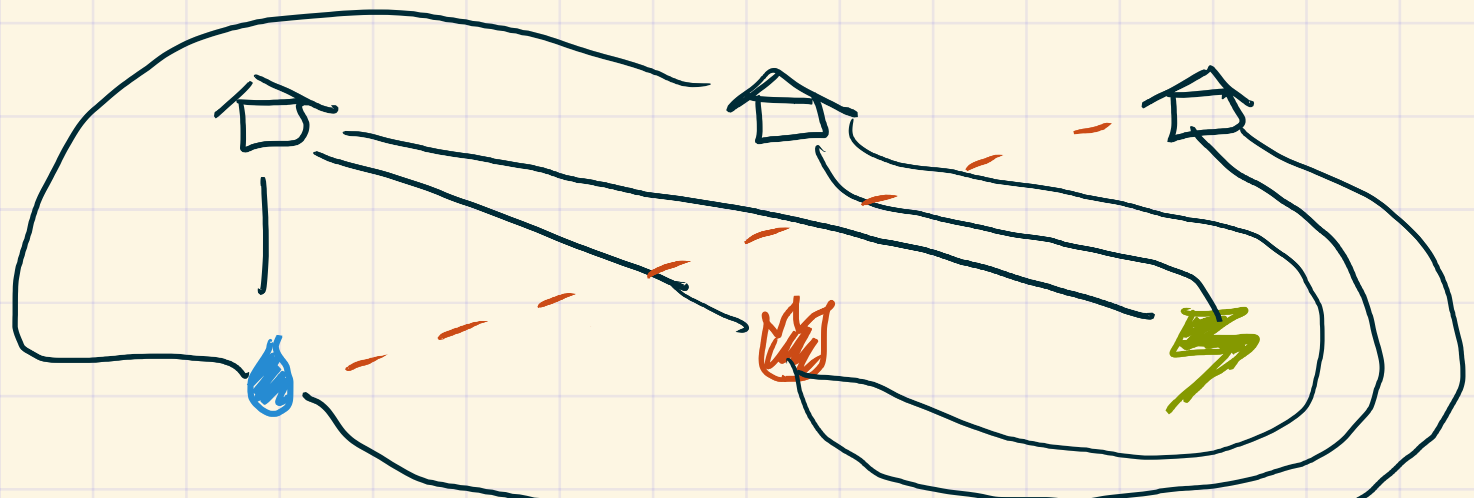
5) Hamiltonian path?
Can you go through each vertex of the graph exactly once?

6) Travelling salesman problem?
What is the shortest circuit that visits each vertex at least once?

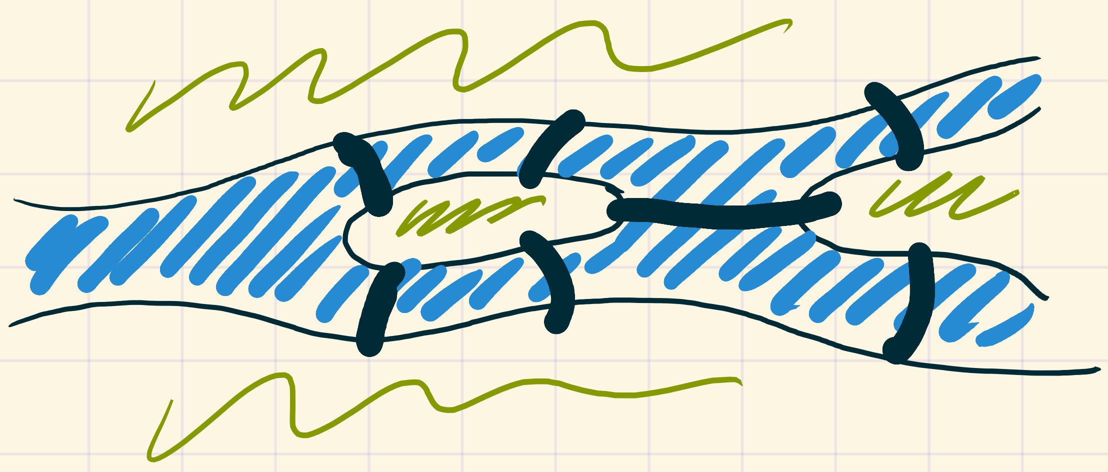
7) Eulerian path problem?
Can you go through each edge of the graph exactly once?

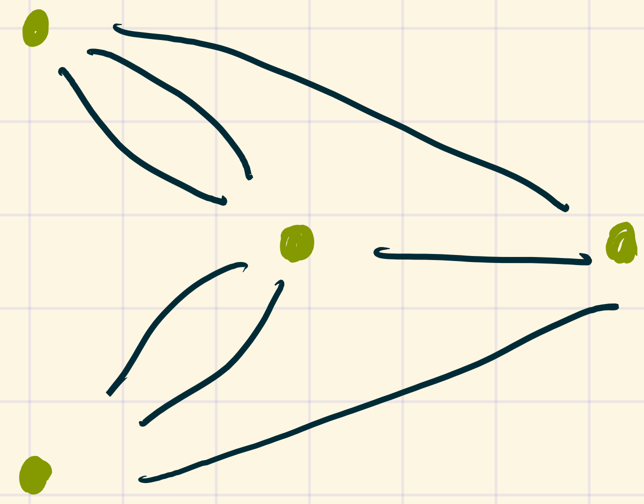
8) Three cottages problem: connect edges without overlaps?

(Planarity.)



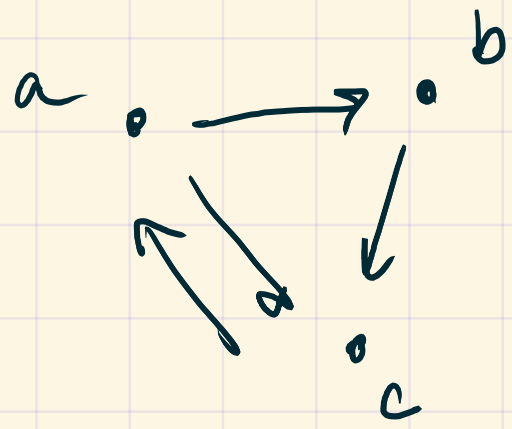
Königsberg bridge puzzle (bridge layout in Königsberg):





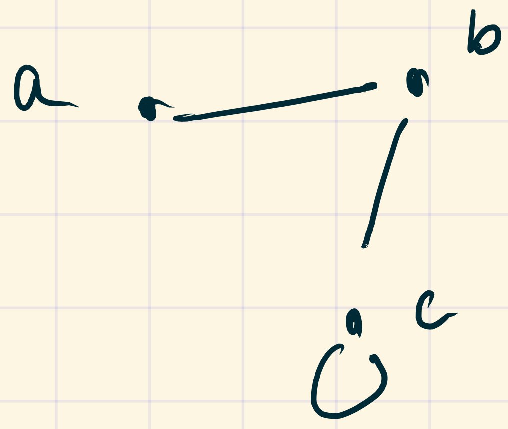
(non-simple)

Adjacency matrix:



$$\begin{array}{c|ccc} & (a) & (b) & (c) \\ \hline (a) & \multicolumn{3}{c}{\multirow{3}{*}{$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$}} \\ (b) \\ (c) \end{array}$$

(directed graph)



$$\begin{array}{c|ccc} & (a) & (b) & (c) \\ \hline (a) & \multicolumn{3}{c}{\multirow{3}{*}{$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$}} \\ (b) \\ (c) \end{array}$$

(undirected graph)

Matrix products:

Let A be an $m \times \underline{n}$ matrix (m rows, n columns)

B be an $\underline{n} \times k$ matrix (n rows, k columns)

Now you can multiply $A \cdot B$

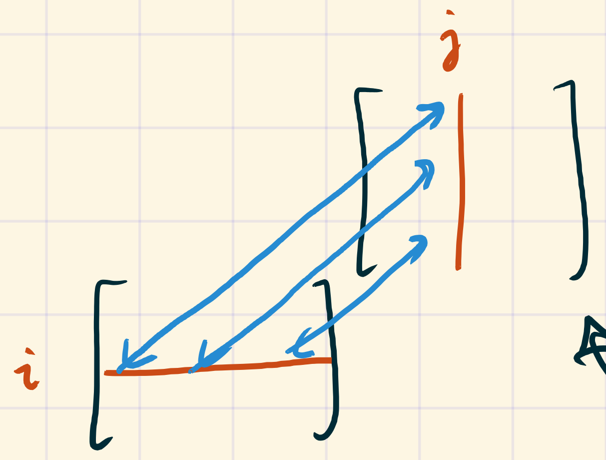
Example

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 3 \end{bmatrix} :$$

$$\begin{array}{cc} & \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 2 \end{matrix}} & \begin{matrix} -1 \\ 0 \end{matrix} & \begin{matrix} 5 \\ 3 \end{matrix} \end{bmatrix} \\ \begin{bmatrix} \boxed{1 \quad 2} \\ 0 \quad 1 \end{bmatrix} & \end{array}$$

$1 \cdot 1 + 2 \cdot 2$
$= 5$

$$= \begin{array}{c|ccc} & (1) & (2) & (3) \\ \hline (1) & \boxed{5} & -1 & 11 \\ (2) & 2 & 0 & 3 \end{array}$$



$(i, j)^{th}$ coordinate of AB

$$
\begin{tikzcd}
a \arrow[r] & b \arrow[dl] \\
c &
\end{tikzcd}
\qquad
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
A \cdot A = A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\begin{array}{c} A^2 \cdot A \\ \parallel \\ A \cdot A^2 \end{array} = A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

* A^2 shows "second level adjacency"!