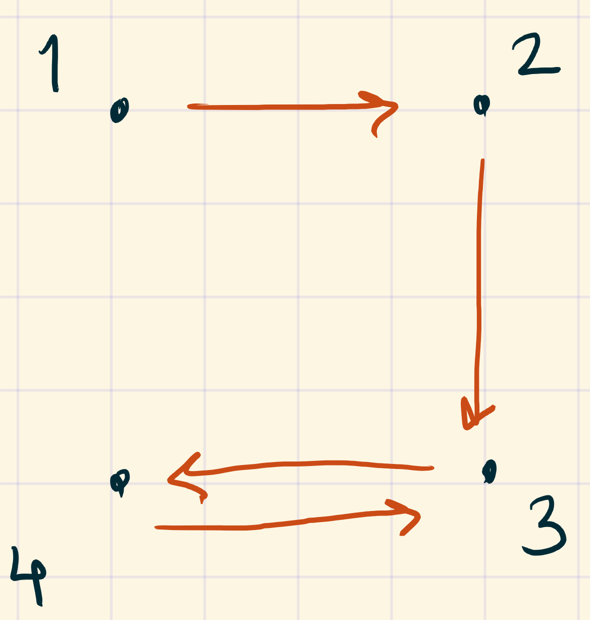
MATH 2301 7 Aug 2020

* Last time : Adjacency matrices

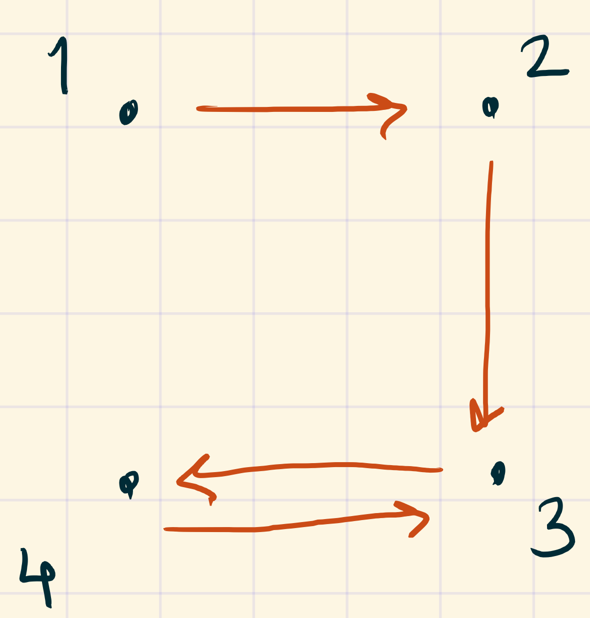
* Today's topic : Computing with adjacency matrices!



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \boxed{0 \quad 1 \quad 0 \quad 0} \\ 0 \quad 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad 1 \\ 0 \quad 0 \quad 1 \quad 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

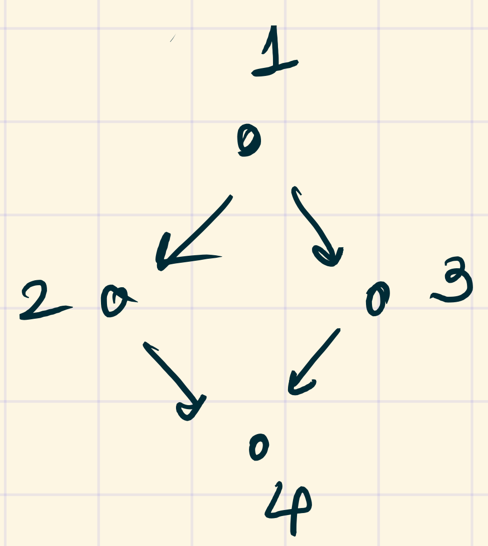
Tip: Ex. In 2^{nd} row, the third entry is 1 & everything else is 0. This third entry will multiply with the elements in the 3^{rd} row of the second copy of A, so in this case we only need to look at the 3^{rd} row of A, multiplied with this entry → to get the 2^{nd} row of the product.



$$A^2 = \begin{array}{c|cccc} & (1) & (2) & (3) & (4) \\ \hline (1) & 0 & 0 & 1 & 0 \\ (2) & 0 & 0 & 0 & 1 \\ (3) & 0 & 0 & 1 & 0 \\ (4) & 0 & 0 & 0 & 1 \end{array}$$

→ Tells you that there is one length-2 path from 1 to 3.

Example :



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

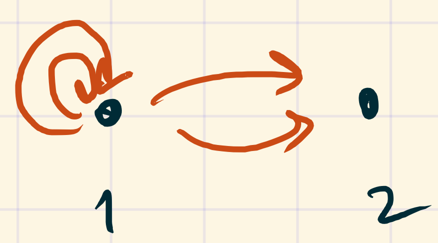
$$= \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underbrace{0 \cdot 0}_{\substack{1 \to 1 \\ 1 \to 4}} + \underbrace{1 \cdot 1}_{\substack{1 \to 2 \\ 2 \to 4}} + \underbrace{1 \cdot 1}_{\substack{1 \to 3 \\ 3 \to 4}} + \underbrace{0 \cdot 0}_{\substack{1 \to 4 \\ 4 \to 4}}$$

<u>Thm</u>: The $(i,j)^{th}$ entry of A^k is exactly the number of length-k paths from i to j

$$i \; \underbrace{\bullet \to \bullet \to \bullet \to \cdots \to \bullet}_{k \text{ edges}} \; j$$

$\ast$ <u>Aside</u>: If you had a non-simple graph:



In the $(i,j)^{th}$ entry of A, put the number of edges from i to j

$$A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$$

(Variant of the adjacency matrix, but the theorem still works!)

<u>Rmk</u>: We can now answer the question of finding length-k paths between vertices.

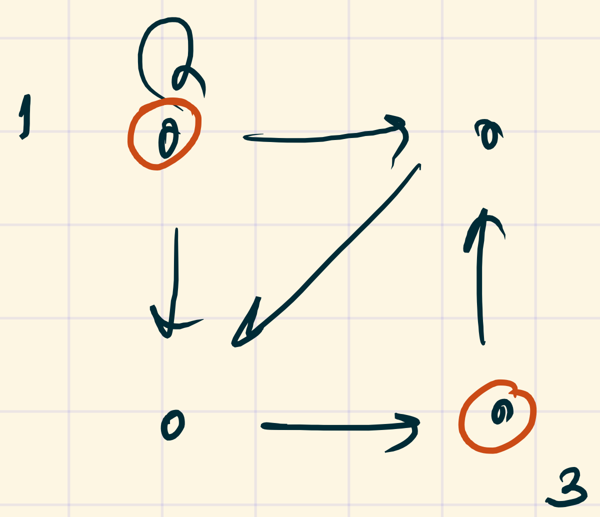
* Connectedness ?

Q: Is there always at least one path from i to j, of some length? (For some given vertices i & j?)

Ans You can look at the $(i,j)^{th}$ entry of

$(A + A^2 + A^3 + \cdots \quad)$ add corresponding entries,

& if this entry ever becomes non-zero, then you know the answer is yes.

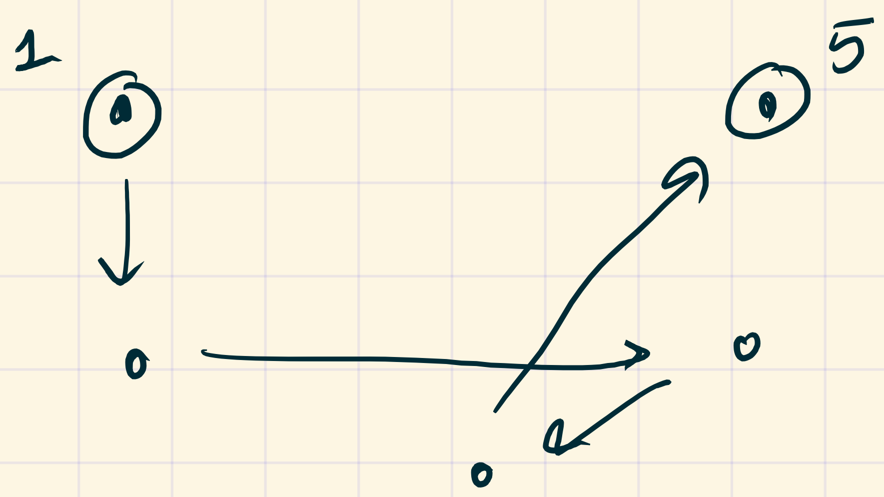


If the graph has n vertices, and if there is some path from i to j, you're guaranteed at least one path of length at most $(n-1)$.

$(A + A^2 + A^3 + \cdots + A^{n-1})$ ← In this sum, you can stop after A^{n-1}.

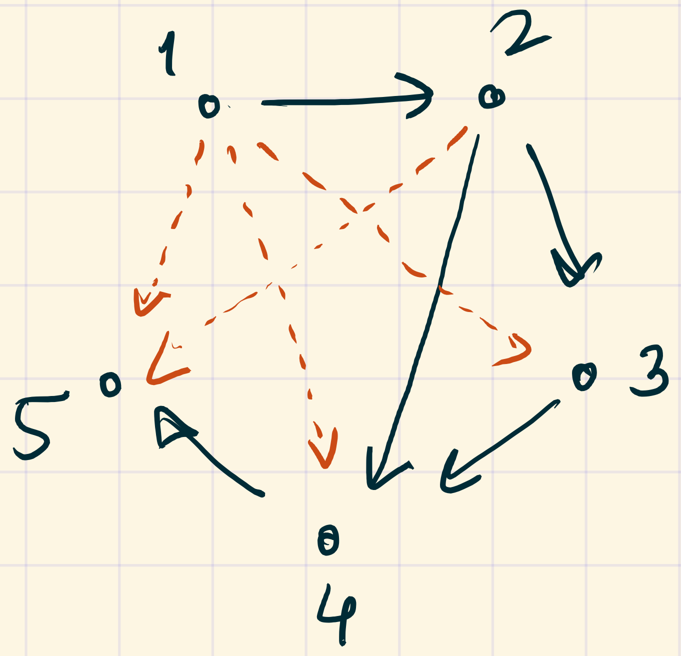
If you don't see a path from i to j until A^{n-1}, you will never see one.

This is because the longest possible length of the shortest path from i to j is $(n-1)$



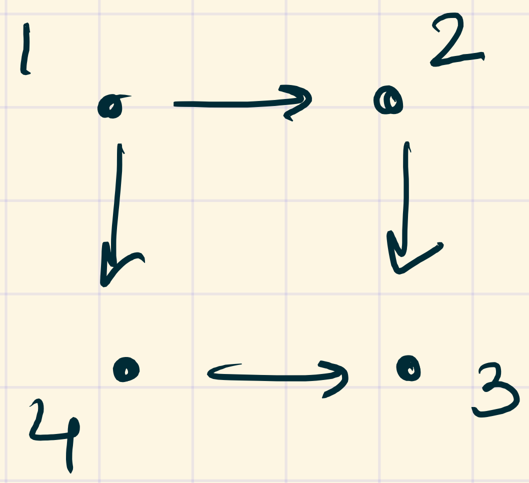
Exercise: Do some examples!

Transitive closure of a relation?



Transitive closure obtained by adding the dotted edges.

In terms of matrices: $\nearrow$ (1,3) needs to be an edge.



(the graph of a relation)

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

there is a length 2 path from 1 to 3!

For transitive closure: If you get a positive entry in A^k at spot (i, j), then add an edge from i to j.

Algebraically: We encode this procedure by "boolean matrix product."

$\longrightarrow$ Next Wednesday