MATH 2301 13 August 2020

* Recap : Boolean products & transitive closure

Thm : The adjacency matrix of the transitive closure of a relation can be computed as

(Component-wise "or" addition)

$$A \vee A^{*2} \vee A^{*3} \cdots \vee A^{*n}$$

(n = # of vertices)

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* Aside : Fast(er) matrix multiplication.

[Use that matrix multiplication / Boolean matrix multiplication is associative : $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ & $A * (B * C) = (A * B) * C$]

* Technique of repeated squaring

Example : To compute $A^4 = ((A \cdot A) \cdot A) \cdot A$, ← 3 products

rewrite as $(A \cdot A) \cdot (A \cdot A)$

① Compute $B = A^2$, then $B^2 = A^4$ ← 2 products

Example : $A^{17} = A \cdot A^{16}$

$A^{16} = A^8 \cdot A^8 = (A^8)^2$

$A^8 = A^4 \cdot A^4 = (A^4)^2$

$A^4 = A^2 \cdot A^2 = (A^2)^2$

$A^2 = (A)^2$.

5 product operations

① Compute $B_1 = A^2$

② Compute $B_2 = B_1^2 = A^4$

③ Compute $B_3 = B_2^2 = A^8$

④ Compute $B_4 = B_3^2 = A^{16}$

⑤ Compute $A \cdot B_4 = A^{17}$

More traditionally :

$A^{17} = (A \cdot A) \cdot A \cdots A$

16 product operations !

$$A^{25} = A \cdot A^{24} = A \cdot (A^{12})^2 = A \cdot ((A^6)^2)^2$$

$$= A \cdot (((A^3)^2)^2)^2$$

$A^3 = A \cdot (A^2)$ ... What are we doing ??

Secretly, we're writing the exponent (25) in binary.

That is, write it as a sum of distinct powers of 2

$$25 = 16 + 8 + 1 = 2^4 + 2^3 + 1$$

$$\parallel$$

$$1 + 24 = 1 + 2(12) = 1 + 2(2(6)) = 1 + 2(2(2(3)))$$

$$= 1 + 2(2(2(1+2))) = 1 + 8 + 16$$

[We'll encounter binary writing again later...]

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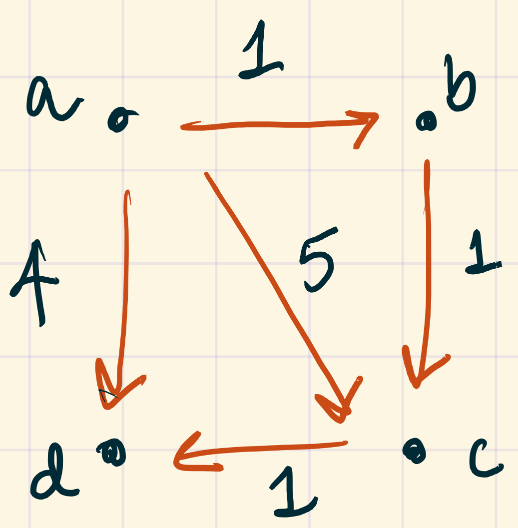
* Another variation on matrix products.

* Weighted graphs

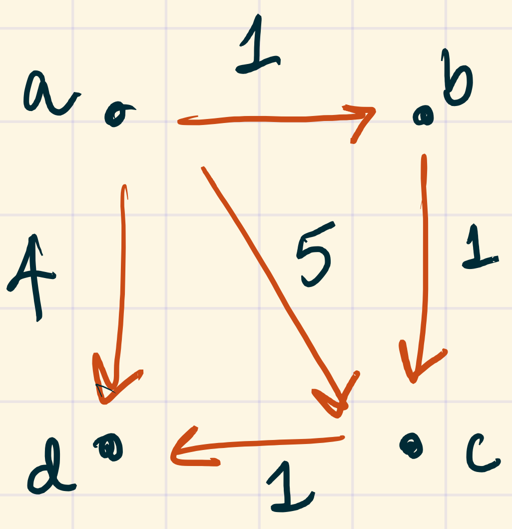


Graphs enhanced with a "weight" for each edge. This is a number attached to each edge, which can represent the length, cost etc depending on your situation.

E.g. Consider a road network with path lengths.

Problem: Find a path from i to j that has lowest weight. (e.g. shortest distance)

* Weighted adjacency matrix



In $(i,j)^{th}$ spot, put the weight of the edge $i \to j$.

[What if there is no edge ??] Definition.

$$\begin{bmatrix} 0 & 1 & 5 & 4 \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 0 \end{bmatrix} = W$$

If $i \neq j$ & there is no edge $i \to j$, let's put "∞"

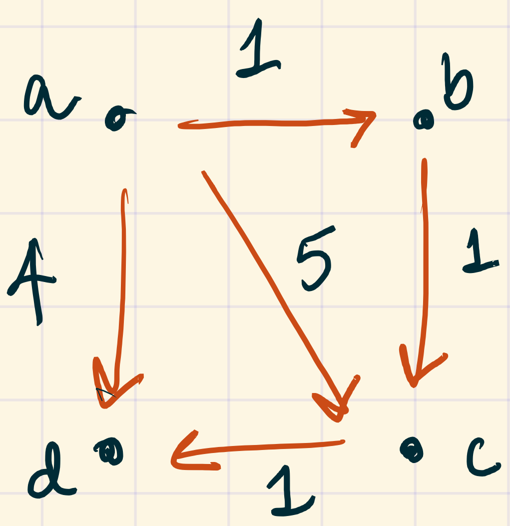
If $i = j$, then we put 0 because there is a path of weight / length 0.

Note: This is not the only choice you can make ... other choices will have other implications

* But what is ∞ ?? In our context, it's just a placeholder: it represents a number so large that it always wins in any comparison we make in our calculations.

$$\max\{\infty, r\} = \infty \qquad [\text{if } r \text{ is a real number}]$$

& $\infty + r = \infty$

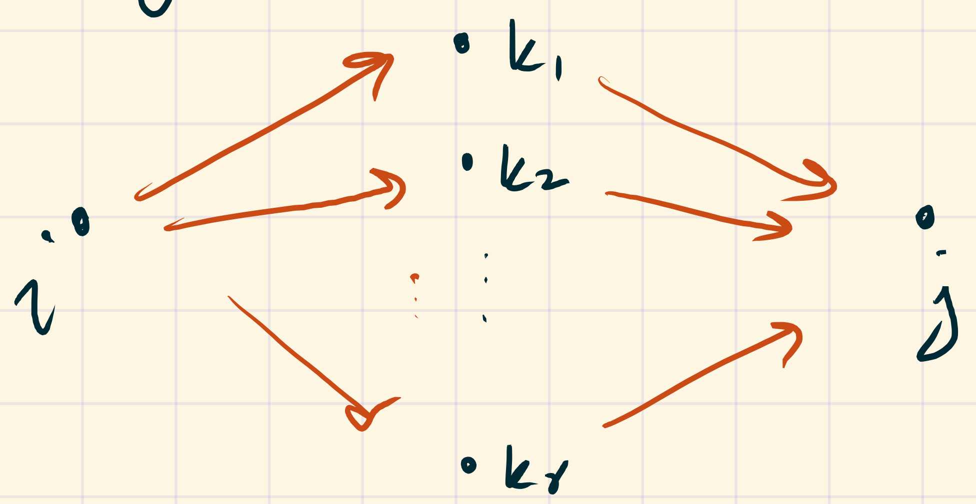


Let's find the least-cost path from any i to any j, using a variant matrix product on the weighted adjacency matrix

E.g.

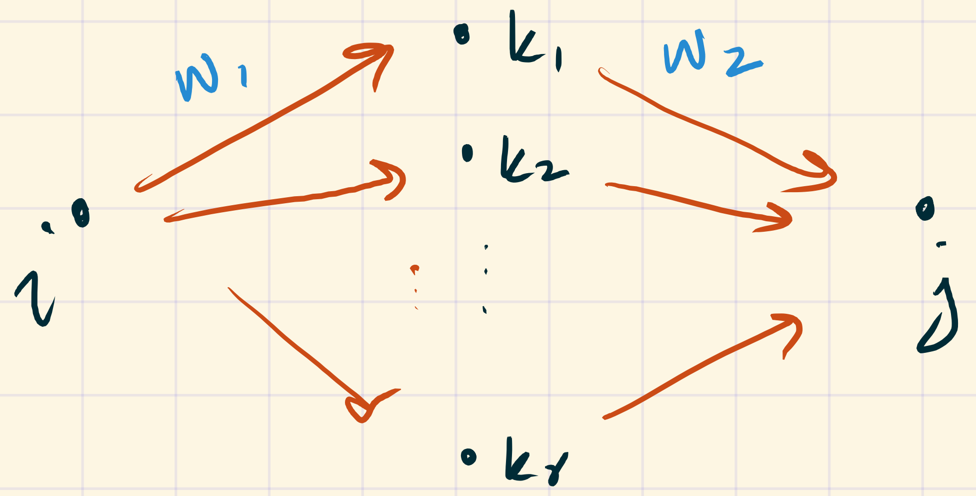
$$W = \begin{bmatrix} 0 & 1 & 5 & 4 \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 0 \end{bmatrix}$$

Typically in A^2, the $(i,j)^{th}$ entry is a combination:



} counts them.

In our setup:



We want to compute the total wt of each such path [e.g. $w_1 + w_2$] & then take the minimum value.

→ We need to take a "min, +" product!

Rules:

① Replace "+" by "min"

② Replace "." by "+"

$$W = \begin{bmatrix} 0 & 1 & 5 & 4 \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 0 \end{bmatrix}$$

The $(1,4)^{th}$ entry of $W \odot W$?

$$\min\{0+4,\ 1+\infty,\ 5+1,\ 4+0\}$$
$$= \min\{4, \infty, 6, 4\}$$
$$= 4.$$

Note: This is the cost of the shortest path from a to d of length ≤ 2.

Thm: The $(i,j)^{th}$ entry of $W^{\odot n}$ gives you minimum cost path from i to j that has at most n edges.

$$W = \begin{bmatrix} 0 & 1 & 5 & 4 \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 0 \end{bmatrix}.$$

$(1,1)^{th}$ entry $= \min\{0+0,\ 1+\infty,\ 5+\infty,\ 4+\infty\} = 0$

$(1,2)^{th}$ entry $= \min\{0+1,\ 1+0,\ 5+\infty,\ 4+\infty\} = 1$

$(1,3)^{th}$ entry $= \min\{0+5,\ 1+1,\ 5+0,\ 4+\infty\} = 2$