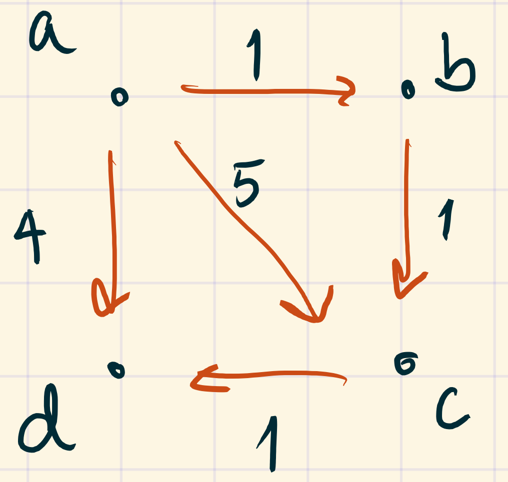
MATH 2301 14 August 2020

* Current topic : shortest paths in weighted graphs.



$$W = \begin{bmatrix} 0 & 1 & 5 & 4 \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 0 \end{bmatrix}$$

(0s on diagonal is a convention, ∞ is a convention)

The "min, +" matrix product $W \odot W$ is like the usual product, but with

① "+" replaced by "min"

② "$\cdot$" replaced by "+"

If

$$W = \begin{bmatrix} 0 & 1 & 5 & 4 \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 0 \end{bmatrix},$$

we have the following powers:

Least-cost paths with ≤ 2 edges

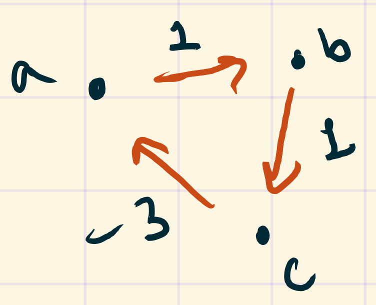
Least cost paths with ≤ 3 edges

$$W \odot W = \begin{bmatrix} 0 & 1 & 2 & 4 \\ \infty & 0 & 1 & 2 \\ \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 0 \end{bmatrix} \qquad W^{\odot 3} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ \infty & 0 & 1 & 2 \\ \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 0 \end{bmatrix}$$

Thm: If the graph has n vertices, then

① $W^{\odot k}$ gives the min-cost paths with $\leq k$ edges between pairs of vertices

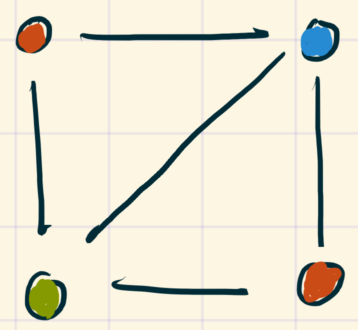
② $W^{\odot (n-1)}$ gives the min-cost paths of any length between pairs of vertices [assuming weights are ≥ 0]



(Neg weight example)

* Graph colouring [undirected graphs without loops Q]

Let $G = (V, E)$ be an undirected graph w/o loops.



Proper 3-colouring

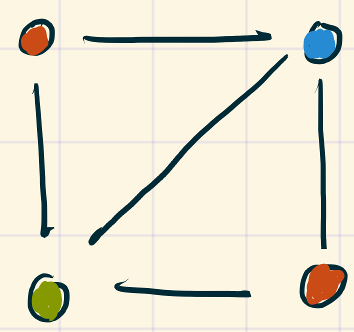
Defn: A (proper) k-colouring of this graph is an assignment of each vertex to one of k colours, such that if $i - j$ then v_i & v_j have different colours.

If $k \geq |V|$ then clearly there is a (proper) k-colouring

number of vertices

Defn: Let $G = (V, E)$ be a graph. The lowest k for which there is a proper k-colouring of G is called the chromatic number of G, denoted $\gamma(G)$

gamma

Example  has chromatic number 3.

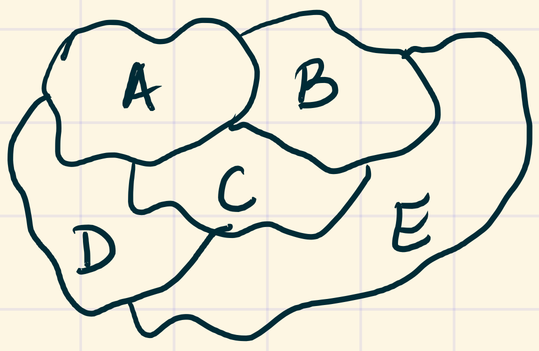
Example (of a colouring)  [4 colours available]

$\gamma(G) = 2$ b/c the graph has a proper 2-colouring, but not a proper 1-colouring.

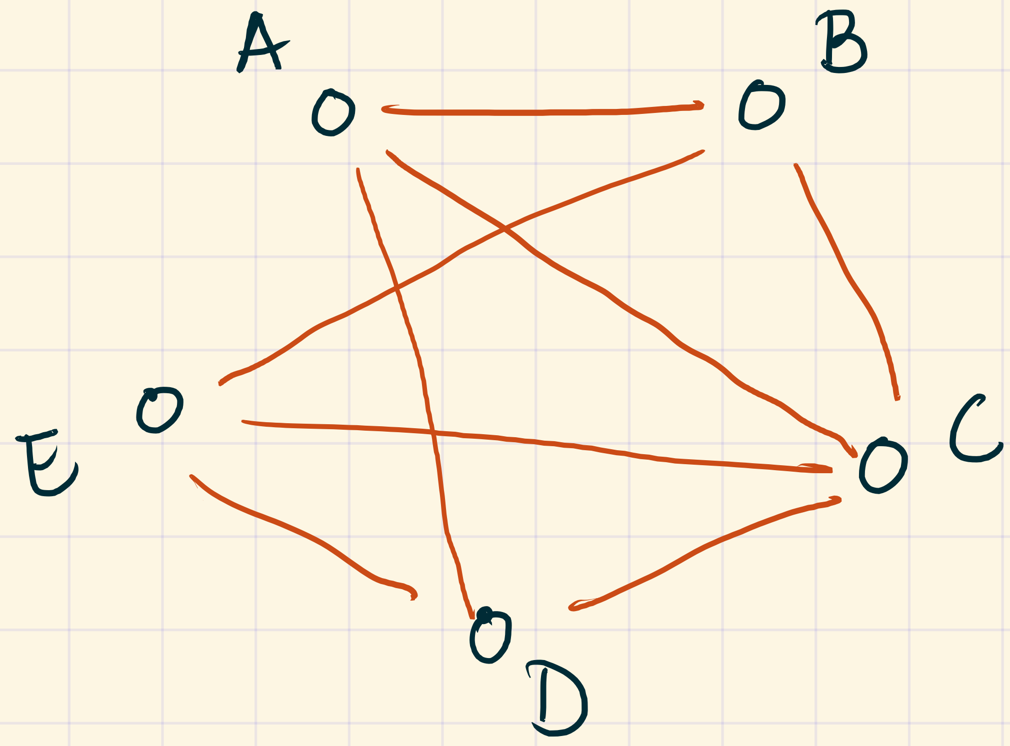
Q: The four-colour problem?

Consider a map. Can you colour it in 4 colours, so that adjacent regions have different colours?

* Relationship with graph colouring?



$\longleftrightarrow$

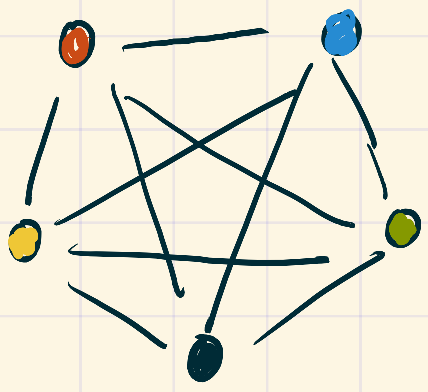


"Dual graph of this map"

A vertex for each region, an edge (i, j) whenever i & j share a border.

[Come back to this later]

Q: Can you come up with a graph whose chromatic number is 5?



$\leftarrow$ Cannot be properly coloured with 4 colours!

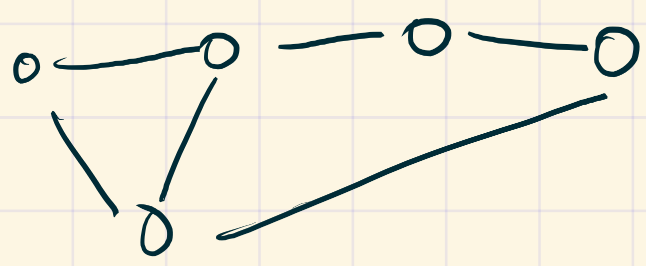
$\leftarrow$ 5-clique.

How about $\chi(G) = 2020$?

Defn: G is called an n-clique if it has n vertices & each vertex is connected to every other vertex.

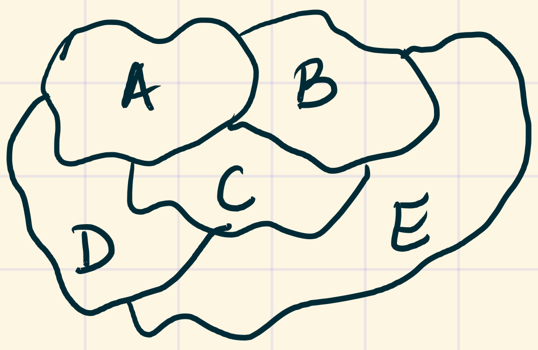
Prop: ① If G is an n-clique, then $\chi(G) = n$.

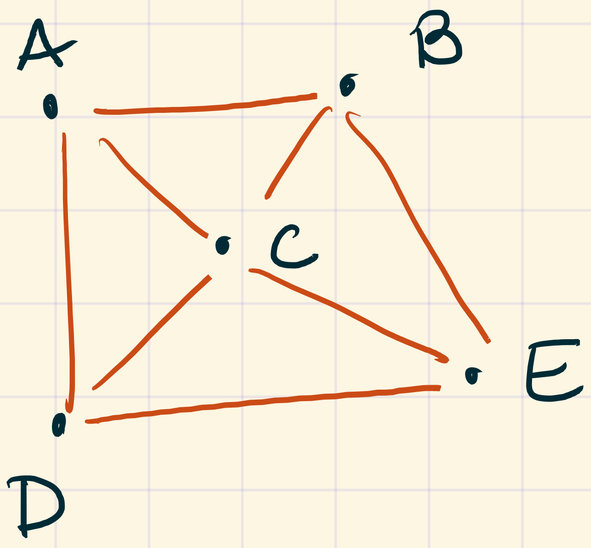
② If G has an n-clique as a subgraph then $\chi(G) \geq n$.



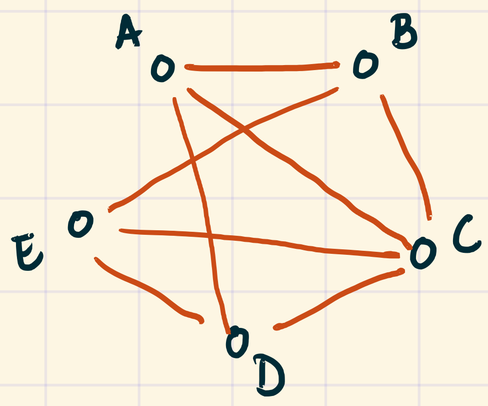
$\leftarrow \chi(G) \geq 3$.

Defn: $H = (W, F)$ is called a subgraph of $G = (V, E)$ if $W \subseteq V$ & $F \subseteq E$ (every vertex of H is a vertex of G & every edge of H is also an edge of G)





In this way of drawing, the edges don't cross!



Edges AD & BE cross

<u>Note</u>: Every dual graph of any map on the plane can be drawn in such a way that edges don't cross! A graph that has this property is called a <u>planar</u> graph.

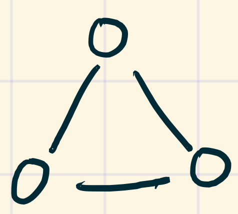
<u>Q</u>: (The four-colour problem): Is every planar graph 4-colourable?

<u>Thm</u> (Four-colour theorem): YES!

Proved in 1970s with

---

* <u>Chromatic function</u>:



<u>Defn</u>: The chromatic function of G is defined as follows:

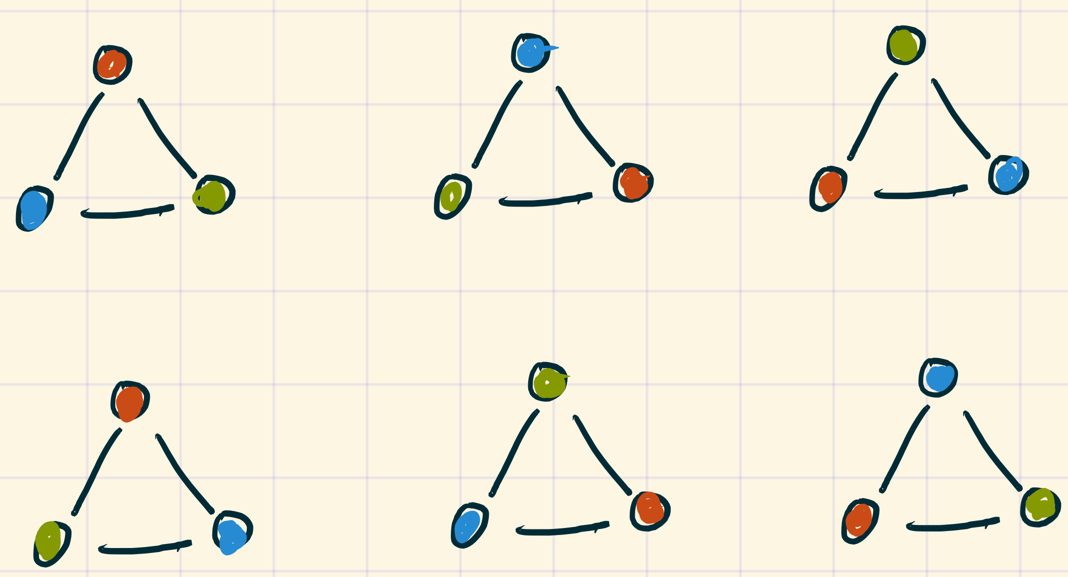
$P(G, k)$ = number of different proper k-colourings of G.

$P(G, 1) = 0$

$P(G, 2) = 0$

$P(G, 3) = 6$.

$P(G, 4) = ??$



These are all the ways to 3-colour