


* Last time: Graph colouring. We learned about:

- Proper k -colourings (k available colours)
- Planar graphs & the 4-colour theorem
[proved by Appel-Haken in 1976; the 5-colour theorem for planar graphs had been known since the late 1800s!]
- The chromatic number $\chi(G)$ of a graph
- n -cliques and their effect on the chromatic number (complete n -graph)
(If a graph G has an n -clique as a subgraph, then $\chi(G) \geq n$.)

- The chromatic function $P(G, t)$ of a graph.

$P(G, t)$ = number of t -colourings of G .

Ex:  = G , $P(G, 1) = 0$, $P(G, 2) = 0$, $P(G, 3) = 6$, $P(G, 4) = ?$ (work-sheet)

Ex: $0-0-0$ $P(G, 1) = 0$, $P(G, 2) = 2$ [R-B-R or B-R-B]

$P(G, 3) = 3! + \binom{3}{2} \cdot 2 = 3 \times 2 + \frac{3 \times 2}{2} \times 2 = 12$

if we use all 3 \uparrow number of ways to choose 2 things out of 3

$P(G, 4) = \binom{4}{3} \cdot 3! + \binom{4}{2} \cdot 2 = 4 \cdot 3! + \frac{4 \times 3}{2} \cdot 2 = 36$

using 3 colours out of 4 \uparrow using 2 colours out of 4.

$0-0-0$

What is $P(G, t)$?
(A general formula?)

- \rightarrow If we use exactly 3 colours out of t , we get $\binom{t}{3} \cdot 3! = \frac{t(t-1)(t-2)}{3!} \cdot 3! = t(t-1)(t-2)$
- \rightarrow If we use exactly 2 colours out of t , we get $\binom{t}{2} \cdot 2 = \frac{t(t-1)}{2} \cdot 2 = t(t-1)$

All together, $P(G, t) = t(t-1)(t-2) + t(t-1) = t(t-1)(t-2+1) = t(t-1)^2$

* Note that if $n < m$, then $\binom{n}{m} = 0$.

* Sanity check: $P(G, 1) = 1(1-1)^2 = 0$
 $P(G, 2) = 2(2-1)^2 = 2$
 $P(G, 3) = 3(3-1)^2 = 3 \cdot 4 = 12$
 $P(G, 4) = 4(4-1)^2 = 4(9) = 36$

Q: If G is any given graph, do we expect a nice formula for $P(G, t)$? And if yes, how can we compute it?

Somewhat surprising answer: Yes!

Thm: If G is any ^(finite) undirected graph without _{self} loops, then $P(G, t)$ is a polynomial in t !

[ie it looks like $a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$]

In fact $P(G, t)$ is also called the chromatic polynomial.

We'll come back to this after looking at partial orders.

* Aside: The Hadwiger-Nelson problem.
 [aka the "chromatic number of the plane" problem].

Consider the real plane \mathbb{R}^2 .

Construct a graph where $V = \mathbb{R}^2$,

$$\text{and } E = \left\{ (p, q) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid d(p, q) = 1 \right\}$$

$(p_1, p_2) \quad (q_1, q_2)$

E.g.



unit circle around a point.

Q: What is the chromatic number of this graph?

Answer?

① This number is finite. In fact it is ≤ 7 .
 [see Wikipedia article on Hadwiger-Nelson problem.]

② This number is ≥ 4 .
 [see Wikipedia; there are finite subgraphs with chromatic number 4.]

Unsolved problem: What is the actual number??

Recent progress: (2018): de Grey proved that
 $\chi(G) \geq 5$!

* We'll pause graph theory for now...
 there are lots of exciting avenues/problems in graph theory.

* Partial orders: a special type of relation; which is reflexive, transitive, and anti-symmetric