MATH 2301 19 August 2020

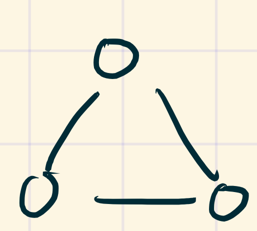
* Last time : Graph colouring. We learned about:

- Proper k-colourings (k available colours)
- Planar graphs & the 4-colour theorem

  [proved by Appel-Haken in 1976; the 5-colour theorem for planar graphs had been known since the late 1800s!]
- The chromatic number $\gamma(G)$ of a graph
- n-cliques (complete n-graph) and their effect on the chromatic number

  (If a graph G has an n-clique as a subgraph, then $\gamma(G) \geq n$.)
- The chromatic function $P(G,t)$ of a graph.

  $P(G,t)$ = number of t-colourings of G.

Ex:  $= G$,

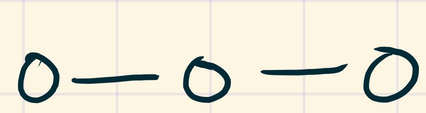
$P(G,1) = 0$

$P(G,2) = 0$

$P(G,3) = 6$

$P(G,4) \stackrel{?}{=}$

$P(G,t) \stackrel{?}{=}$ [worksheet]

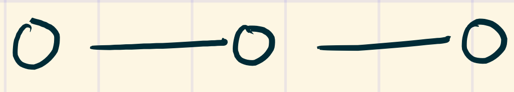
Ex: 

$P(G,1) = 0$

$P(G,2) = 2$ [R-B-R or B-R-B]

$$P(G,3) = \underset{\text{if we use all 3}}{3!} + \underset{\substack{\text{number of ways to choose}\\ \text{2 things out of 3}}}{\binom{3}{2}} \cdot 2 = 3\times 2 + \frac{3\times 2}{2}\times 2 = 12$$

$$P(G,4) = \underbrace{\binom{4}{3}\cdot 3!}_{\text{using 3 colours out of 4}} + \underbrace{\binom{4}{2}\cdot 2}_{\text{using 2 colours out of 4.}} = 4\cdot 3! + \frac{4\times 3}{2}\cdot 2 = 36$$



What is $P(G,t)$?
(A general formula?)

→ If we use exactly 3 colours out of t,
we get $\binom{t}{3} \cdot 3! = \frac{t(t-1)(t-2)}{3!} \cdot 3! = \boxed{t(t-1)(t-2)}$

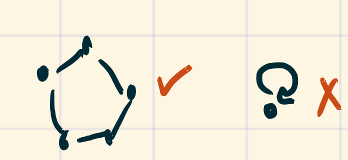
→ If we use exactly 2 colours out of t,
we get $\binom{t}{2} \cdot 2 = \frac{t(t-1)}{2} \cdot 2 = \boxed{t(t-1)}$

All together, $P(G,t) = t(t-1)(t-2) + t(t-1)$
$= t(t-1)(t-2+1) = t(t-1)^2$

* Note that if $n < m$, then $\binom{n}{m} = 0$.

* Sanity check: $P(G,1) = 1(1-1)^2 = 0$
$P(G,2) = 2(2-1)^2 = 2$
$P(G,3) = 3(3-1)^2 = 3 \cdot 4 = 12$
$P(G,4) = 4(4-1)^2 = 4(9) = 36$.

Q: If G is any given graph, do we expect a nice formula for $P(G,t)$? And if yes, how can we compute it?



Somewhat surprising answer: Yes!

Thm: If G is any (finite) undirected graph without self loops, then $P(G,t)$ is a polynomial in t!

[i.e it looks like $a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$]

In fact $P(G,t)$ is also called the chromatic polynomial.

We'll come back to this after looking at partial orders.

* <u>Aside</u> : The Hadwiger-Nelson problem.

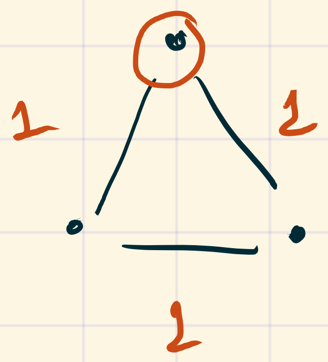
[aka the "chromatic number of the plane" problem].

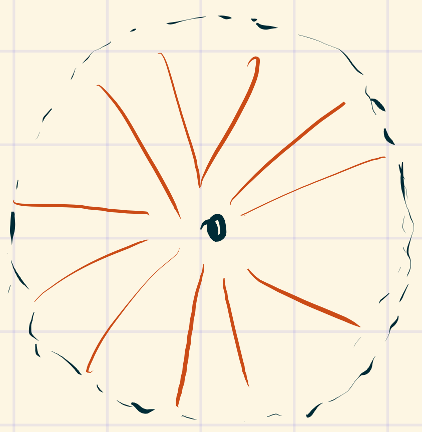
Consider the real plane $\mathbb{R}^2$.

Construct a graph where $V = \mathbb{R}^2$,

and $E = \{ (\underset{(p_1,p_2)}{p}, \underset{(q_1,q_2)}{q}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid d(p,q) = 1 \}$

<u>E.g.</u>





unit circle around a point.

<u>Q</u>: What is the chromatic number of this graph?

<u>Answer</u>?

① This number is finite. In fact it is ≤ 7.
[see Wikipedia article on Hadwiger-Nelson problem.]

② This number is ≥ 4.
[see Wikipedia; there are finite subgraphs with chromatic number 4.]

<u>Unsolved problem</u> : What is the actual number??

<u>Recent progress</u> : (2018) : de Gray proved that

$\chi(G) \geq 5$!

* We'll pause graph theory for now...

there are lots of exciting avenues/problems in graph theory

* <u>Partial orders</u> : a special type of relation; which is reflexive, transitive, and anti-symmetric