

* Finished (most of) the section on graph theory

* Today: Partial orders.

Defn: A relation R on a set S is called a partial order if it satisfies

① Reflexivity, ② Anti-symmetry, and ③ Transitivity.

[Gives you a (partial) hierarchy among elements of your set S]

* Notation: If R is a partial order on S ,

① S is called a partially ordered set or poset

② If $(a, b) \in R$, then we'll write

$a \leq_R b$ "a is before b" in this hierarchy.
or $a \leq b$

* This relation need not be numerical

③ If $a \leq_R b$ and $a \neq b$, then we write

$a <_R b$ or $a < b$.

* We might also write $b \geq a$ if we know
 $a \leq b$.

Examples

① R on $S = \mathbb{N}$, given by

$$R = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \leq b\}.$$

Check: ① Reflexivity: b/c $a \leq a \forall a$

② Anti-symmetry: b/c if $a \leq b$ & $b \leq a$ then $a = b$

③ Transitivity: b/c if $a \leq b$ & $b \leq c$ then $a \leq c$.

In this case: $a \geq b$ just means $a \leq b$

② $S = \mathbb{N}$, & define

$$R = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \geq b\}.$$

weird but perfectly valid!

In this case: $a \leq_R b$ means $a \geq b$

③ Let S be any set & define a relation R on $\mathcal{P}(S)$, such that

$$R = \{(A, B) \in \mathcal{P}(S) \times \mathcal{P}(S) \mid A \subseteq B\}.$$

$$\text{Specific example: } S = \{1, 2\} \quad \left| \begin{array}{l} R \text{ is on } \mathcal{P}(S) \\ R \subseteq \mathcal{P}(S) \times \mathcal{P}(S) \end{array} \right.$$

$$R = \{(\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\}), (\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{1, 2\}), (\{2\}, \{2\}), (\{2\}, \{1, 2\}), (\{1, 2\}, \{1, 2\})\}$$

Why a partial order?

① Reflexivity: $A \subseteq A \forall A \in \mathcal{P}(S)$

② Anti-symmetry: If $A \subseteq B$ & $B \subseteq A$ then $A = B$

③ Transitivity: If $A \subseteq B$ & $B \subseteq C$ then $A \subseteq C$.

Remark: In examples ① & ②, we have the following property: for any two $a, b \in S$ we had either $a \leq_R b$ or $b \leq_R a$

That is, any two elements were comparable

But Example ③ doesn't have this property.

because $\{1\} \not\subseteq \{2\}$ and $\{2\} \not\subseteq \{1\}$.

That is, $\{1,3\}$ & $\{2,3\}$ are incomparable.

Defn: A partial order \leq on S is called a total order if any two elements of S are comparable.

E.g. ① & ② from earlier

Example

④ Let $S = \mathbb{N}$ [or any subset of \mathbb{N} if you like]

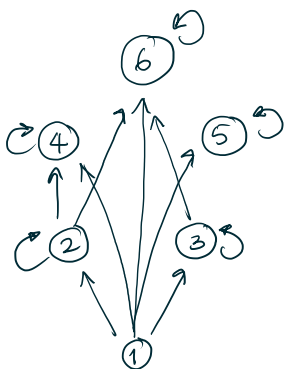
Define $R \subseteq S \times S$ as follows:

$$R = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a|b\} \leftarrow \text{check that this is a partial order.}$$

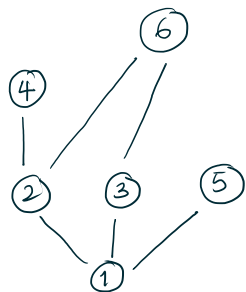
More specific: $S = \{1, 2, 3, 4, 5, 6\}$

Graph:

Cluttered



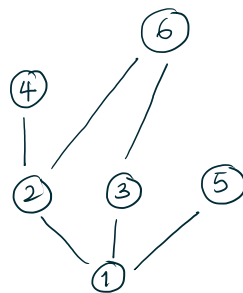
Hasse diagram



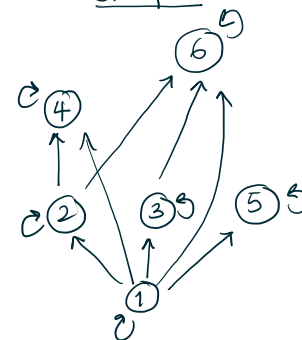
- ① Each edge has an implicit direction from bottom to top
- ② Omit all self-loops (implied by reflexivity)
- ③ Omit all edges implied by transitivity

* The directed graph can be recovered from the Hasse diagram

Hasse diagram

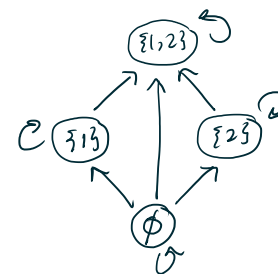
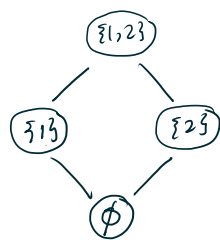


Graph



- Add
- ① Directions
 - ② Reflexive closure
 - ③ Transitive closure.

Example: $R = \text{subset relation for } \mathcal{P}(\{1,2,3\})$.



Example: $S = \{1, 2, 3, 4\}$ with \leq relation

(Linear diagram)



* Observe: If \leq is a total order, then the Hasse diagram is linear as above.

Example ⑤ - Making dinner [pizza + salad + brownies]

Steps:

Pizza

- ① Make dough
- ② Make toppings
- ③ Sauce
- ④ Shred cheese
- ⑤ Top pizza
- ⑥ Bake

Salad

- Chop greens
- Make dressing
- Toppings
- Mix

Brownies

- Make batter
- Prepare pan
- Bake

This forms a poset

- Serve pizza
- Serve salad
- serve brownies

① ≤ ⑤ ≤ ⑥

② ≤ ⑤ ≤ ⑥

③ ≤ ⑤ ≤ ⑥

④ ≤ ⑤ ≤ ⑥

↪ order of tasks

GANTT charts

critical path analysis etc

(souped-up versions of posets)