MATH 2301 20 August 2020

* Finished (most of) the section on graph theory
* Today: Partial orders.

Defn: A relation R on a set S is called a partial order if it satisfies

① Reflexivity, ② Anti-symmetry, and ③ Transitivity.

[Gives you a (partial) hierarchy among elements of your set S]

* Notation: If R is a partial order on S,

① S is called a partially ordered set or poset

② If $(a,b) \in R$, then we'll write

$a \preceq_R b$ "a is before b" in this hierarchy.

or $a \preceq b$

* This relation need not be numerical

③ If $a \preceq_R b$ and $a \neq b$, then we write

$a \prec_R b$ or $a \prec b$.

* We might also write $b \succeq a$ if we know $a \preceq b$.

Examples

① R on $S = \mathbb{N}$, given by

$R = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \leq b\}$.

"for all"

Check: ① Reflexivity: b/c $a \leq a \ \forall a$

② Anti-symmetry: b/c if $a \leq b$ & $b \leq a$ then $a = b$

③ Transitivity: b/c if $a \leq b$ & $b \leq c$ then $a \leq c$.

In this case: $a \preceq b$ just means $a \leq b$

② $S = \mathbb{N}$, & define

$R = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \geq b\}$.

weird but perfectly valid!

In this case: $a \preceq_R b$ means $a \geq b$

③ Let S be any set & define a relation R on $P(S)$, such that

$R = \{(A,B) \in P(S) \times P(S) \mid A \subseteq B\}$.

Specific example: $S = \{1,2\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

R is on $P(S)$

$R \subseteq P(S) \times P(S)$.

$R = \{(\phi, \{1\}), (\phi, \{2\}), (\phi, \{1,2\}), (\phi, \phi)$
$(\{1\}, \{1\}), (\{1\}, \{1,2\}), (\{2\}, \{2\}), (\{2\}, \{1,2\}),$
$(\{1,2\}, \{1,2\})\}$

Why a partial order?

① Reflexivity: $A \subseteq A \;\; \forall \; A \in P(S)$

② Anti-symmetry: If $A \subseteq B$ & $B \subseteq A$ then $A = B$

③ Transitivity: If $A \subseteq B$ & $B \subseteq C$ then $A \subseteq C$.

Remark: In examples ① & ②, we have the following property: for any two $a, b \in S$ we had either $a \preceq_R b$ or $b \preceq_R a$

That is, any two elements were comparable

But Example ③ doesn't have this property because $\{1\} \nsubseteq \{2\}$ and $\{2\} \nsubseteq \{1\}$.

That is, $\{1\}$ & $\{2\}$ are incomparable.

Defn: A partial order on S is called a total order if any two elements of S are comparable.

E.g. ① & ② from earlier

Example

④ Let $S = \mathbb{N}$ [or any subset of $\mathbb{N}$ if you like]

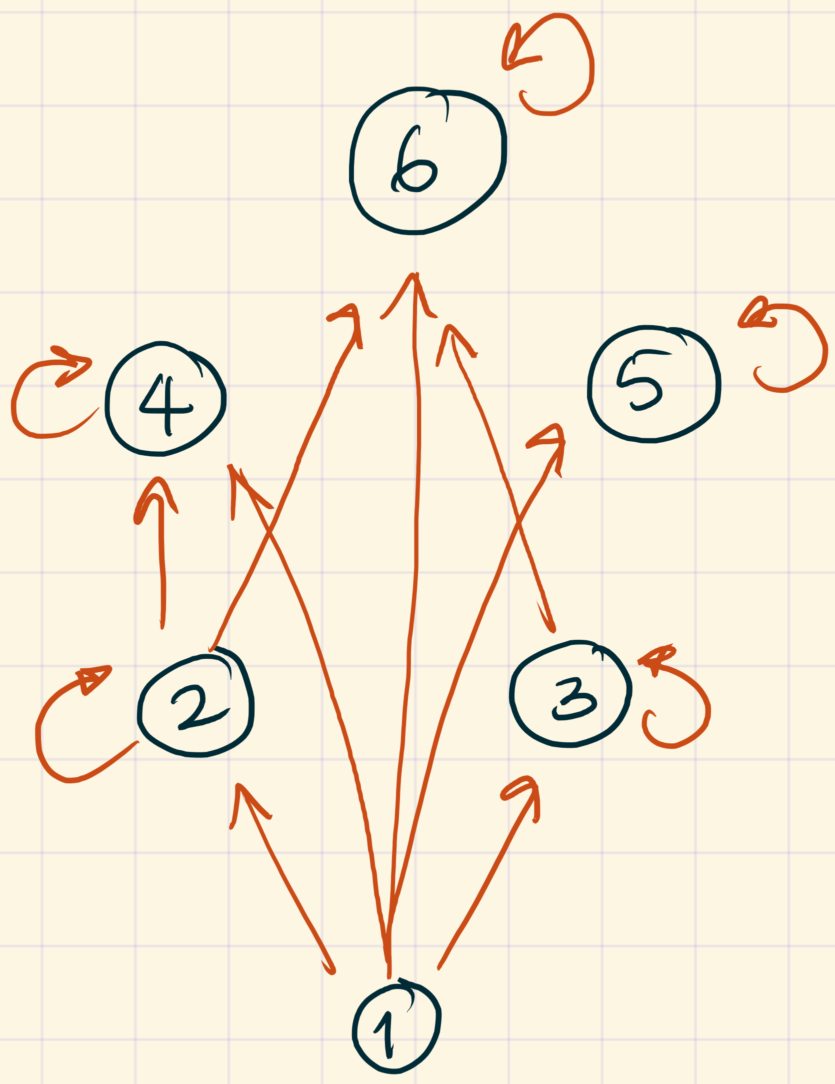
Define $R \subseteq S \times S$ as follows:

$R = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \mid b\}$ ← Check that this is a partial order.

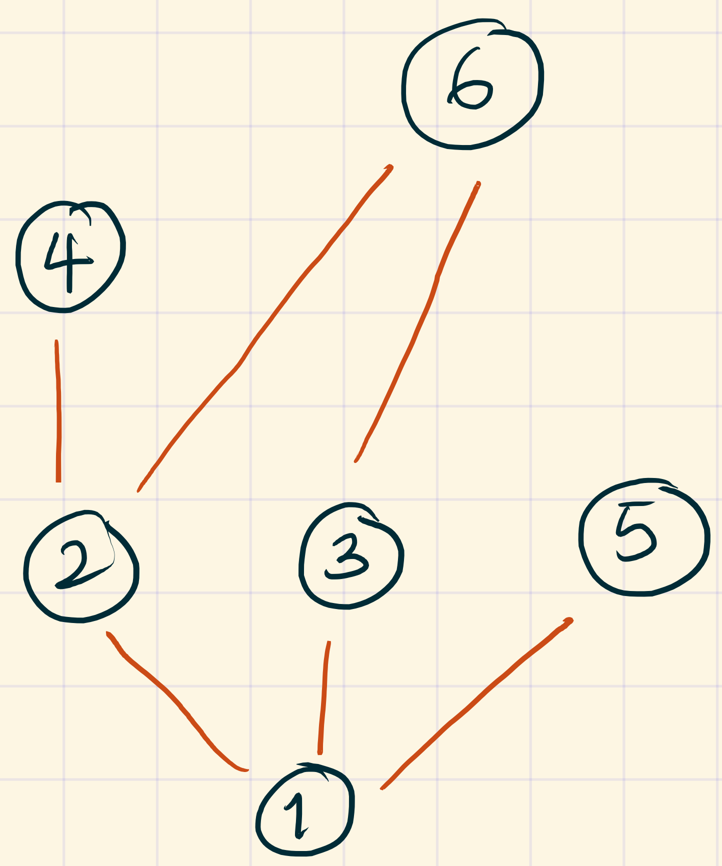
More specific: $S = \{1, 2, 3, 4, 5, 6\}$

Graph:

Cluttered



Hasse diagram



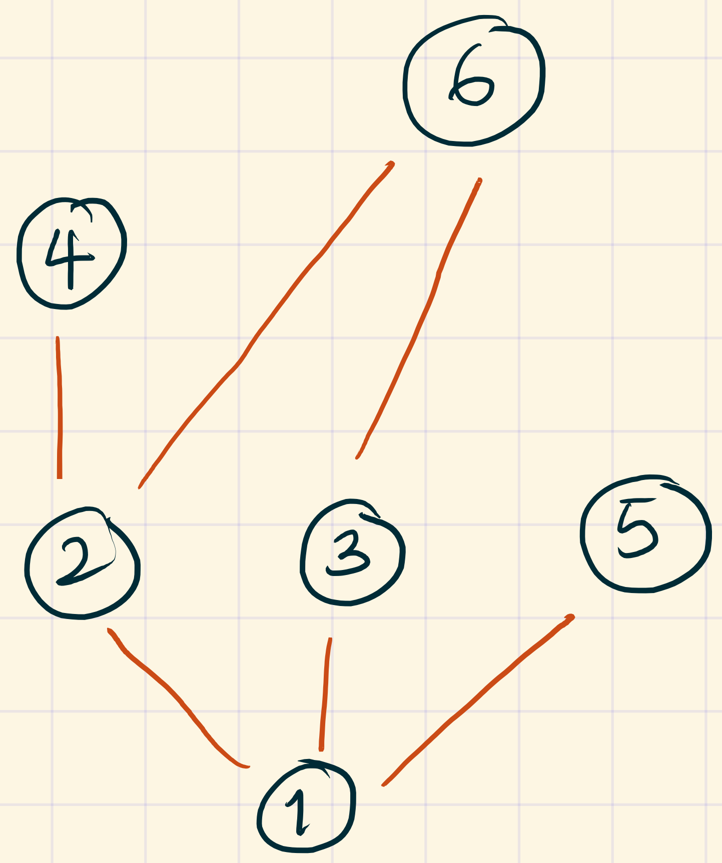
① Each edge has an implicit direction from bottom to top

② Omit all self-loops (implied by reflexivity

③ Omit all edges implied by transitivity

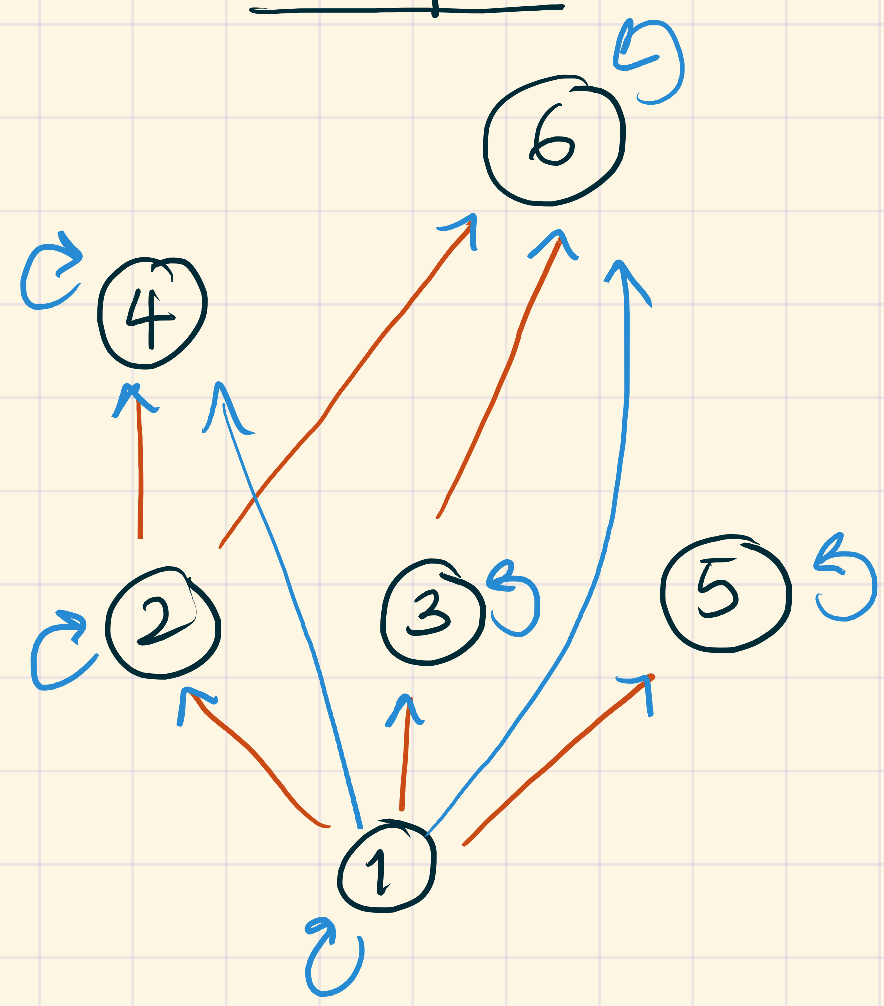
* The directed graph can be recovered from the Hasse diagram

Hasse diagram

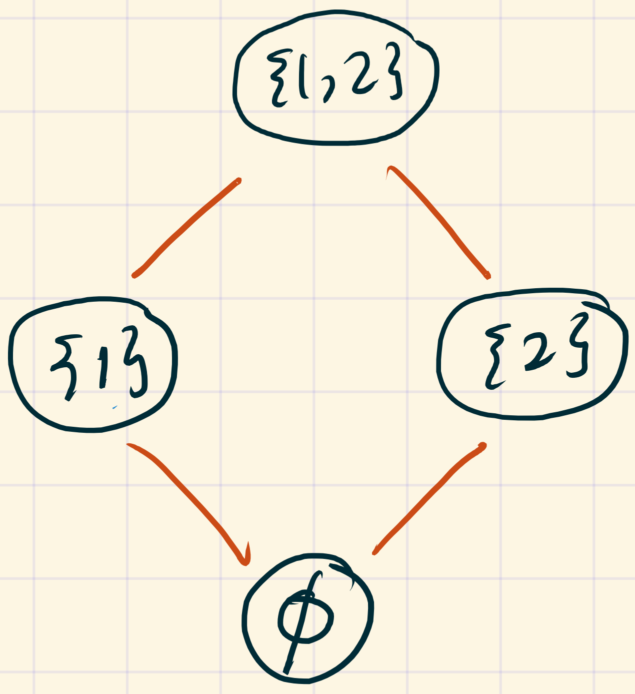


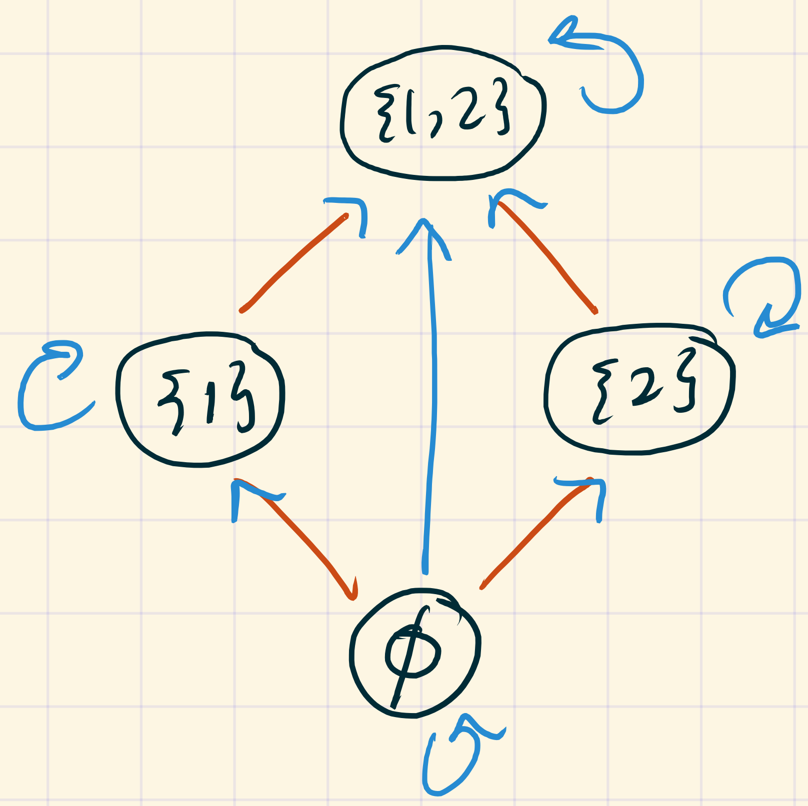
Add
1. Directions
2. Reflexive closure
3. Transitive closure.

Graph



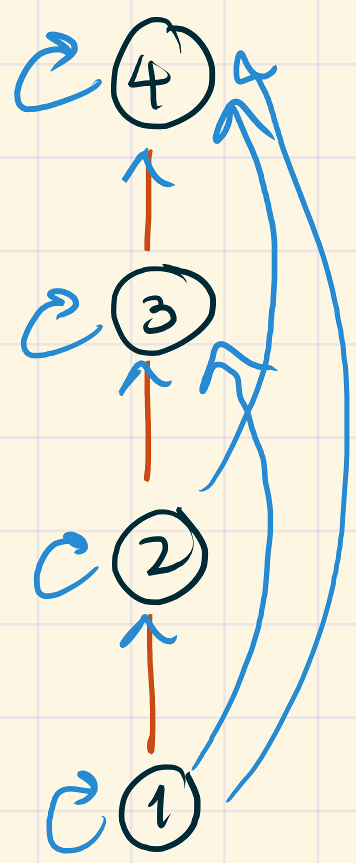
Example : R = subset relation for $\mathcal{P}(\{1,2\})$





Example : $S = \{1,2,3,4\}$ with $\leq$ relation

(Linear diagram)



* Observe : If $\leq$ is a total order, then the Hasse diagram is linear as above.

Example 5. Making dinner [pizza + salad + brownies]

Steps:

Pizza	Salad	Brownies
1. Make dough	- Chop greens	- Make batter
2. Make toppings	- Make dressing	- Prepare pan
3. Sauce	- Toppings	- Bake
4. Shred cheese	- Mix	
5. Top pizza		
6. Bake		

This forms a poset

- Serve pizza
- Serve salad
- Serve brownies

$1 \preceq 5 \preceq 6$

$2 \preceq 5 \preceq 6$

$3 \preceq 5 \preceq 6$

$4 \preceq 5 \preceq 6$

$\rightsquigarrow$ order of tasks

GANTT charts
critical path analysis etc
(souped-up versions of posets)