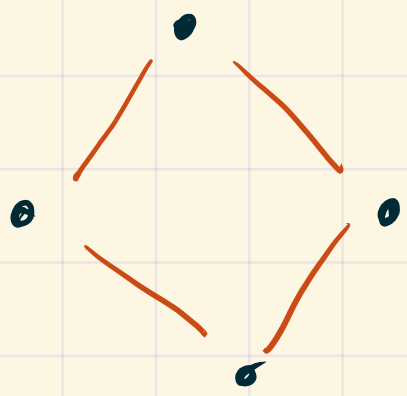
MATH 2301 21 August 2020

* Last time : - We defined partial orders
  - Looked at some examples & Hasse diagrams

* Hasse diagrams

Example (subset poset for $\{1,2\}$):



* A poset consists of a set S, together with a partial order on S

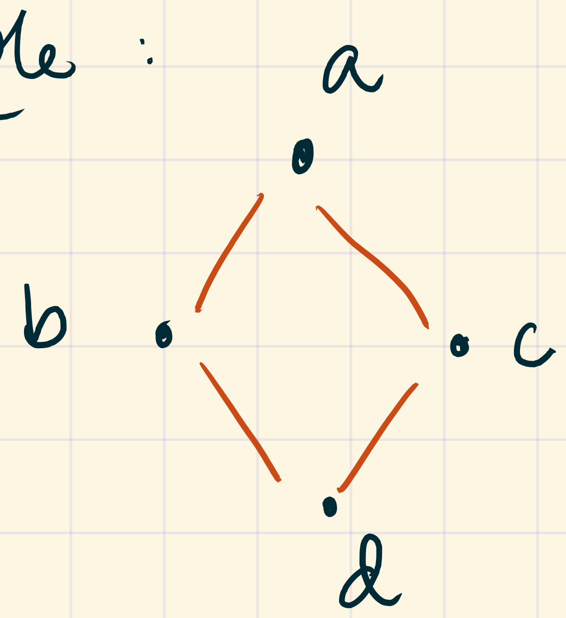
Definition : Two posets are isomorphic if there is an order-preserving bijection from one to the other :

Given $(S_1, \preceq_1)$ & $(S_2, \preceq_2)$, they are isomorphic as posets if we can find some bijective function (one-one & onto)

$f: S_1 \to S_2$ such that
$x \preceq_1 y$ iff $f(x) \preceq_2 f(y)$.

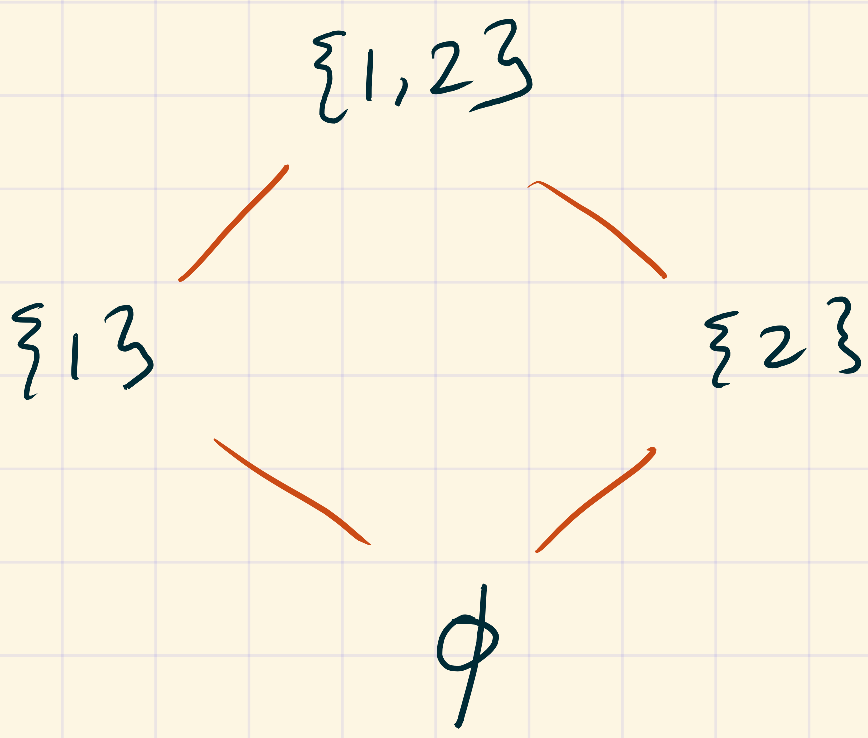
In other words, they have the same Hasse diagram.

Example :



i.e. $d \preceq c$
$d \preceq b$
$b \preceq a$
$c \preceq a$

+ reflexivity & transitivity



For the subset relation

These posets are isomorphic

$\phi \leftrightarrow d$ $\{2\} \leftrightarrow c$
$\{1\} \leftrightarrow b$ $\{1,2\} \leftrightarrow a$

[alternatively, could send $b \leftrightarrow \{2\}$ & $c \leftrightarrow \{1\}$]

Let's draw all possible Hasse diagrams of small size

* 1 element

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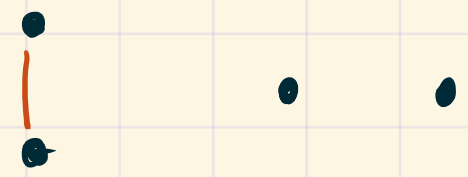
* 2 elements 

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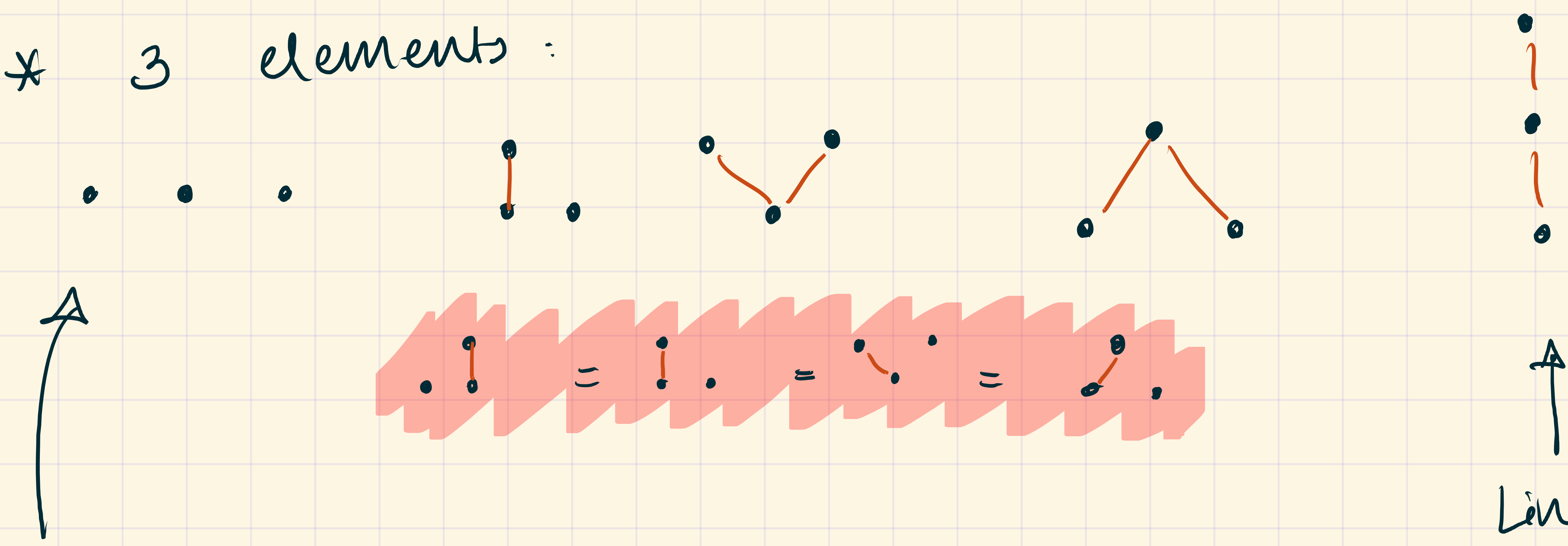
* 3 elements :



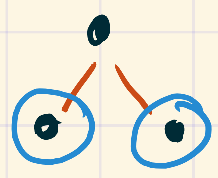
anti-chain

Linear or totally ordered poset, aka a chain

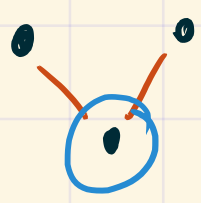
Minimal/minimum & maximal/maximum elements:

Defn: An element x of a poset is called

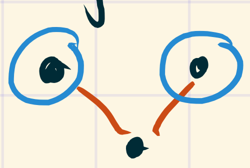
① Minimal if there is no y such that $y \precneqq x$

 ← Minimal but not minimum.

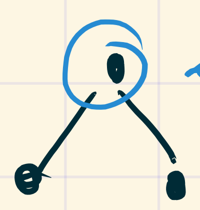
② Minimum if $x \preceq y$ for every $y \in S$.

 ← Minimum element.

③ Maximal if there is no y such that $y \succneqq x$

 ← Maximal but not maximum

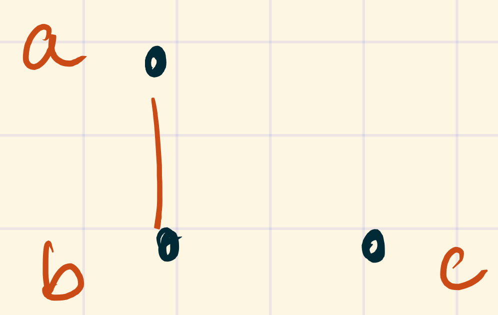
④ Maximum if $x \succeq y$ for every $y \in S$.

 ← Maximum element.

* Note: The minimum element [resp. maximum], if it exists, is unique. Moreover the

minimum [resp. maximum] element is also minimal [resp. maximal].

<u>Example</u>:



No minimum or maximum

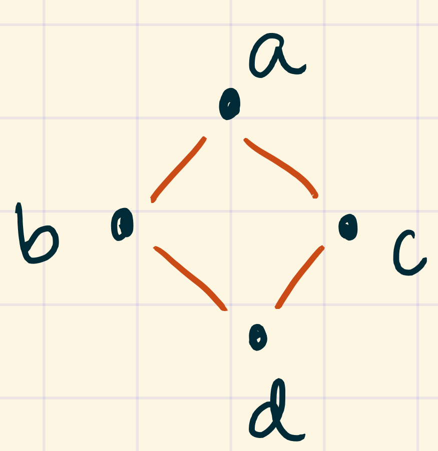
a & c are maximal

b & c are minimal.

<u>Defn</u>: Let $(S, \precsim)$ be a poset. Let $x, y \in S$ such that $x \precsim y$. Then the <u>(closed) interval</u> $[x,y]$ is defined as

$$[x,y] := \{ z \in S \mid x \precsim z \precsim y \}.$$

<u>Example</u>

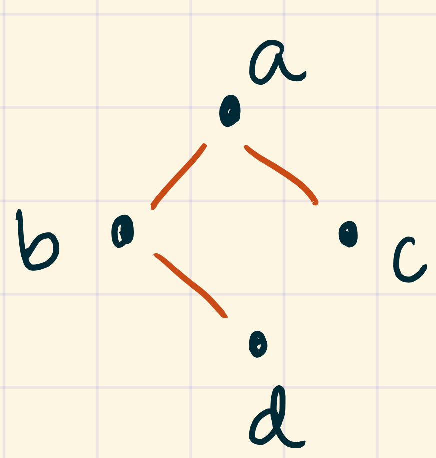


$$[d,a] = \{d, b, c, a\}$$
$$[b,a] = \{b, a\}$$
$$[c,a] = \{c, a\}$$
$$[d,d] = \{d\}$$

<u>Similarly</u>:

$$(x,y) = \{ z \in S \mid x \precnsim z \precnsim y \}$$
$$[x,y) = \{ z \in S \mid x \precsim z \precnsim y \}$$
$$(x,y] = \{ z \in S \mid x \precnsim z \precsim y \}$$

<u>Example</u>:

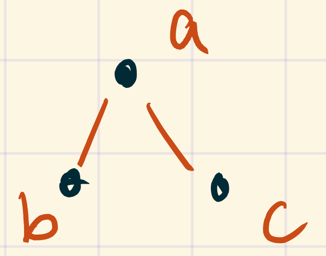


$$[d,a] = \{d, b, a\}$$
$$[a,d] = \emptyset.$$

* Incidence algebra

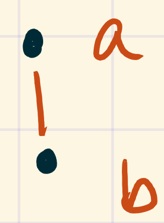
Let $(P, \preceq)$ be a poset.

Let $I(P)$ be the set of all nonempty closed intervals in P

E.g.  $I(P) = \{[b,a], [c,a], [b,b], [c,c], [a,a]\}$

Defn: The incidence algebra $\mathcal{A}_P$ is defined to be the set of all functions from $I(P)$ to $\mathbb{R}$.

$$\mathcal{A}_P := \{ f : I(P) \to \mathbb{R} \}$$

Example:  $I(P) = \{[a,a], [b,b], [b,a]\}$

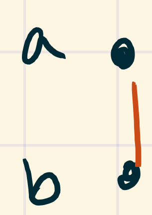
An example element of $\mathcal{A}_P$ is a function that sends
$[a,a] \mapsto 1$
$[b,b] \mapsto 0$
$[b,a] \mapsto 3$

In general, $f \in \mathcal{A}_P$ is a function that assigns a real number to each interval $[x,y]$.

* Examples: Let $(P, \preceq)$ be any (finite) poset.

(zero function) ① $f_0 \in \mathcal{A}_P$ is the function

$$f_0([x,y]) = 0$$



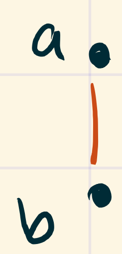
$f_0([a,a]) = 0$
$f_0([b,b]) = 0$
$f_0([b,a]) = 0$

(δ-function) ② $\delta \in \mathcal{A}_P$ is the function

$$\delta([x,y]) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

$\delta([a,a]) = 1$
$\delta([b,b]) = 1$
$\delta([b,a]) = 0$

zeta function ③ $\zeta \in \mathcal{A}_P$ is the function
$\zeta([x,y]) = 1$



$\zeta([a,a]) = \zeta([b,b])$
$= \zeta([b,a]) = 1$

* Things we can do to $\mathcal{A}_p$.

① If $f \in \mathcal{A}_p$, $g \in \mathcal{A}_p$, you can add them:

$$(f+g)([x,y]) := f([x,y]) + g([x,y])$$

② Multiply ??