MATH 2301 26 August 2020

* Some admin : Midterm on Thursday 3 Sep

6:30 pm — 8:30 pm on Zoom

[Link will be released on Wattle shortly before exam.]

[Suggest you arrive by 6:20.]

** Exam structure

- 5 problems, 20 min/problem (solving + scanning) & 5-minute breaks in between
- All problems released simultaneously on Gradescope at exactly 6:30 pm.
- All due by 8:30 pm BUT :
- ** Each problem has a time limit of 20 min (solving + scanning), so

OPEN AND FINISH ONE AT A TIME

** Syllabus

Everything until the end of tomorrow's [27/08] class

** Other info :

- Detailed instructions & policies to be released on Wattle
- You must have a working video setup and use only one device, except perhaps for scanning.
- You must submit on Gradescope. [PDF files]
- No calculators, cheat sheets, or outside material permitted. If you are flagged for suspected cheating, I may request a short oral interview about your exam.

** Questions?

* Recap : - Posets, min/max elements, Hasse diagrams

- A first look at incidence algebras

* Today : More about posets

** GLB & LUB [Greatest lower bound & least upper bound]

Def: Let $(P, \preceq)$ be a poset. Let $S \subseteq P$.
Then an element $x \in P$ is

① an upper bound for S, if $\forall$ (for every) $y \in S$, $y \preceq x$.

② a lower bound for S, if $\forall y \in S$, $y \succeq x$.

Defn: Let $(P, \preceq)$ be a poset & $S \subseteq P$.
Then an element $x \in P$ is

① The LUB for S, if

(a) x is an upper bound for S, and

(b) for any upper bound z of S, we have $x \preceq z$.

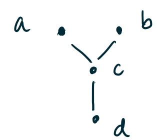
② The GLB of S, if:

(a) x is a lower bound for S, and

(b) for any lower bound z of S, we have $x \succeq z$.

* * Remark : If an LUB exists for a set S, it is unique.
But it need not exist.
Similarly for GLB. — Exercise : verify!

① Examples :



$S_1 = \{a\}$: upper bounds are : a.
in fact, a is the LUB.
Lower bounds : c, d, a.
in fact, a is the GLB.

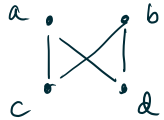
$S_2 = \{a, b\} \to$ No upper bounds, so no LUB.
Lower bounds are c, d & c is the GLB.

$S_3 = \{c, d\} \to$ has c as the LUB and d as the GLB.

② Example : $P = [0, 1)$ with relation $\leq$

Q : What is the LUB of P? $\to$ Does not exist in P.

③ Example :



$S_1 = \{a, b\}$

c & d are both lower bounds

But No GLB.

$S_2 = \{c, d\} \to$ No LUB.

Defn : A poset $(P, \preceq)$ is called a lattice if for any two $a, b \in P$, the subset $\{a, b\}$ has both a LUB and a GLB.

Note : Example 3 is not a lattice.

Exercise : Draw some example lattices.

E.g.  is lattice.

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* Incidence algebras. Let $(P, \preceq)$ be a poset

$I(P)$ = set of closed intervals in P.

A_P = set of functions $f : I(P) \to \mathbb{R}$.

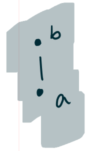
E.g.  $I(P) = \{[a, a], [a, b], [b, b]\}$

Elements in A_P send closed intervals to real numbers.

Three examples

① $f_0 \in A_P$: $f_0([a, a]) = f_0([b, b]) = f_0([a, b]) = 0$.

(works for any poset ; set $f_0([x, y]) = 0$)



② $\delta \in A_P$: $\delta([a, a]) = \delta([b, b]) = 1$, $\delta([a, b]) = 0$

("Kronecker delta function", works for any poset with $\delta([x, y]) = 0$ if $x \neq y$, $\delta([x, x]) = 1$).

③ $\overset{\text{zeta}}{\zeta} \in A_P$: $\zeta([a, a]) = \zeta([a, b]) = \zeta([b, b]) = 1$.

(works for any poset with $\zeta([x, y]) = 1$).

A_P [the incidence algebra] has nice operations.

* Addition : If $f, g \in A_P$, then you can write $(f+g) \in A_P$, defined as

$$(f+g)([x, y]) := f([x, y]) + g([x, y])$$

(just like usual function addition)

E.g. $\underbrace{(\zeta+\zeta)}_{2}([x, y]) = 2$ also written 2ζ.

$(\zeta+\delta)([x, y]) = \begin{cases} 2 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$

* Scalar multiplication

If $f \in A_P$ and $r \in \mathbb{R}$, then we define $(rf) \in A_P$ by : (scalar product)

$$(rf)([x, y]) = r \cdot f([x, y]).$$

normal product of numbers

E.g. $(2\zeta)([x, y]) = 2 \cdot \zeta([x, y]) = \zeta([x, y]) + \zeta([x, y])$

$= (\zeta + \zeta)([x, y])$

$$(f+f+f)([x, y]) = (3f)([x, y])$$

* <u>Multiplication of elements in $\mathcal{A}_P$</u>

A bit complicated! Uses properties of our poset heavily.

Called the "convolution product".