MATH 2301 26 August 2020

* Some admin : Midterm on Thursday 3 Sep
6:30 pm — 8:30 pm on Zoom

[Link will be released on Wattle shortly before exam.]

[Suggest you arrive by 6:20.]

** Exam structure

- 5 problems, 20 min/problem solving + scanning & 5-minute breaks in between
- All problems released simultaneously on Gradescope at exactly 6:30 pm.
- All due by 8:30 pm BUT:
- ** Each problem has a time limit of 20 min solving + scanning, so

OPEN AND FINISH ONE AT A TIME

** Syllabus

Everything until the end of tomorrow's [27/08] class

** Other info :

- Detailed instructions & policies to be released on Wattle
- You must have a working video setup and use only one device, except perhaps for scanning.
- You must submit on Gradescope. [PDF files]
- No calculators, cheat sheets, or outside material permitted. If you are flagged for suspected cheating, I may request a short oral interview about your exam.

** Questions?

* <u>Recap</u> :- Posets, min/max elements, Hasse diagrams
- A first look at incidence algebras

* <u>Today</u> : More about posets.

** <u>GLB & LUB</u> [Greatest lower bound & least upper bound]

<u>Def</u>: Let $(P, \preceq)$ be a poset. Let $S \subseteq P$.
Then an element $x \in P$ is

for every

① an <u>upper bound</u> for S, if $\forall\, y \in S$, $y \succeq x$.

② a <u>lower bound</u> for S, if $\forall\, y \in S$, $y \succeq x$.

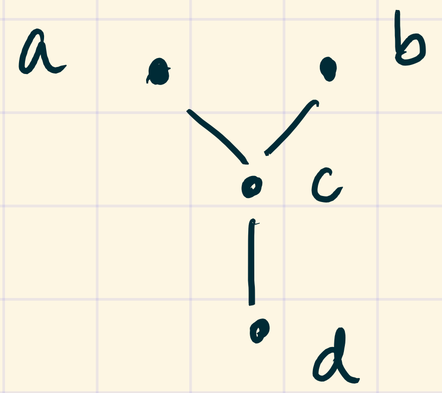
<u>Defn</u>: Let $(P, \preceq)$ be a poset & $S \subseteq P$.
Then an element $x \in P$ is

① The LUB for S, if
(a) x is an upper bound for S, and
(b) for any upper bound z of S, we have $x \preceq z$.

② The GLB of S, if:
ⓐ x is a lower bound for S, and
ⓑ for any lower bound z of S, we have $x \succeq z$.

* * <u>Remark</u>: If an LUB exists for a set S, it is <u>unique</u>.
But it need not exist.
Similarly for GLB. — <u>Exercise</u>: verify!

① <u>Examples</u>:



$S_1 = \{a\}$: upper bounds are : a.
in fact, a is the LUB.
lower bounds : c, d, a.
in fact, a is the GLB.

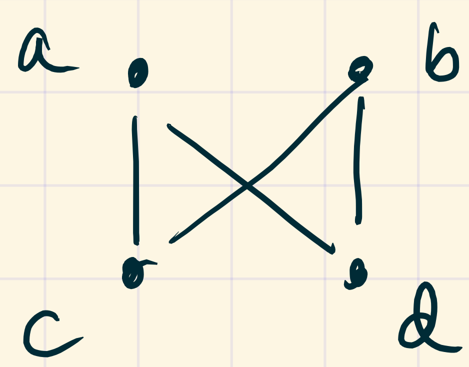
$S_2 = \{a, b\} \rightarrow$ No upper bounds, so no LUB.
Lower bounds are c, d & c is the GLB.

$S_3 = \{c, d\} \to$ has c as the LUB and
d as the GLB.

② Example : $P = [0, 1)$ with relation $\leq$

Q: What is the LUB of P? $\to$ Does not exist in P.

③ Example :



$S_1 = \{a, b\}$

c & d are both lower bounds

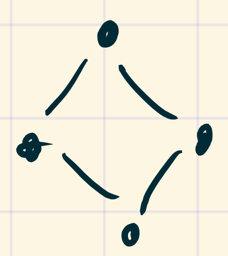
But No GLB.

$S_2 = \{c, d\} \to$ No LUB.

Defn: A poset $(P, \preceq)$ is called a lattice if for any two $a, b \in P$, the subset $\{a, b\}$ has both a LUB and a GLB.

Note: Example 3 is not a lattice.

Exercise: Draw some example lattices.

E.g.  is lattice.

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* Incidence algebras. Let $(P, \preceq)$ be a poset

$I(P)$ = set of closed intervals in P.

A_P = set of functions $f : I(P) \to \mathbb{R}$.

E.g. 

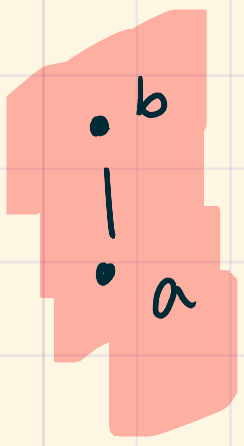
$I(P) = \{[a, a], [a, b], [b, b]\}$

Elements in A_P send closed intervals to real numbers.

Three examples

① $f_0 \in A_P$: $f_0([a, a]) = f_0([b, b]) = f_0([a, b]) = 0$.

(works for any poset ; set $f_0([x, y]) = 0$)



② $\delta \in \mathcal{A}_P$: $\delta([a,a]) = \delta([b,b]) = 1$, $\delta([a,b]) = 0$

("Kronecker delta function", works for any poset with $\delta([x,y]) = 0$ if $x \neq y$, $\delta([x,x]) = 1$).

zeta

③ $\zeta \in \mathcal{A}_P$: $\zeta([a,a]) = \zeta([a,b]) = \zeta([b,b]) = 1$.

(works for any poset with $\zeta([x,y]) = 1$).

$\mathcal{A}_P$ [the incidence algebra] has nice operations.

* Addition: If $f, g \in \mathcal{A}_P$, then you can write $(f+g) \in \mathcal{A}_P$, defined as

$$(f+g)([x,y]) := f([x,y]) + g([x,y])$$

(just like usual function addition)

Eg. $(\zeta+\zeta)([x,y]) = 2$

also written 2ζ.

$(\zeta+\delta)([x,y]) = \begin{cases} 2 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$

* Scalar multiplication

If $f \in \mathcal{A}_P$ and $r \in \mathbb{R}$, then we define

(scalar product) $(rf) \in \mathcal{A}_P$ by:

normal product of numbers

$$(rf)([x,y]) = r \cdot f([x,y]).$$

Eg. $(2\zeta)([x,y]) = 2 \cdot \zeta([x,y]) = \zeta([x,y]) + \zeta([x,y])$

$= (\zeta+\zeta)([x,y])$

$$(f+f+f)([x,y]) = (3f)([x,y])$$

* Multiplication of elements in $\mathcal{A}_P$

A bit complicated! Uses properties of our poset heavily.

Called the "convolution product".