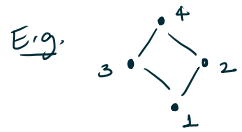* Admin : Mid-term at 6:30pm on 3rd Sep.

Practice questions to be posted today. Typed answers will not be posted, but you are encouraged to discuss with me, each other, & demonstrators. Come to office hour!

* Last time : Addition + scalar multiplication in the incidence algebra
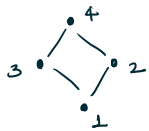
* Today : Convolution product.

E.g.



is $(P, \leq)$.   Recall: $I(P) =$ set of closed intervals of P. [in this case, $I(P)$ has 9 elements]

Definition : Let $(P, \leq)$ be a poset. Let $f, g \in A_P$.
Their convolution product $(f*g)$ is another element of $A_P$, defined as follows: on any interval $[x,y]$

$$(f*g)([x,y]) := \sum_{x \leq z \leq y} f([x,z]) \cdot g([z,y])$$

Warning : $(f*g)([x,y]) \neq f([x,y]) \cdot g([x,y])$
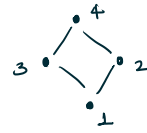
Example : $f([x,y]) := x+y$



i.e. $f([1,3]) = 4$
       $f([1,4]) = 5$

$g([x,y]) = x-y$.

$(f*g)([1,1]) \overset{?}{=} \sum_{1 \leq z \leq 1} f([1,z]) \cdot g([z,1])$     ← z=1 is the only option

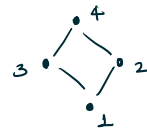$= f([1,1]) \cdot g([1,1]) = (1+1)(1-1) = 0.$

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$(f*g)([1,4]) = \sum_{1 \leq z \leq 4} f([1,z]) \, g([z,4])$     ← there are 4 possibilities for z



$= f([1,1]) \cdot g([1,4]) + f([1,2]) \cdot g([2,4])$

$+ f([1,3]) \, g([3,4]) + f([1,4]) \cdot g([4,4])$

$= 2(-3) + 3(-2) + 4(-1) + 5(0)$

$= -6 + -6 + -4 = -16.$

Exercise : Compute the value of $f*g$ on all 9 intervals.

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* Why this convolution product ??



① Order the vertices of your poset P.
Usually, we try to order "from bottom to top" so as to go with the flow of $\leq$
$(1, 2, 3, 4)$ , for example.
But  $(1, 3, 2, 4)$ is an equally good ordering with respect to this poset.

* These are nice orderings because whenever $a \geq b$ in P, we have a appearing before b.

This is called a topological sorting of P.

Defn : Let $(P, \leq)$ be a poset. An ordering $(p_1, \cdots, p_n)$ of the elements of P is called a topological sorting if whenever $p_i \geq p_j$, we have $i \leq j$.

* Warning : You might have $i < j$ with $p_i \& p_j$ incomparable.

* (Explanation of convolution product contd.)

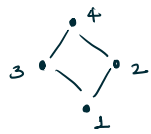① Consider a topological sort of $P$ : $(p_1, \cdots, p_n)$

E.g $(1,2,3,4)$

② Given any $f \in A_P$, we'll write down an $n \times n$ matrix $M_f$, defined as follows.

$$\text{size of } P \downarrow$$

The $(i,j)^{th}$ entry of $M_f$ is:

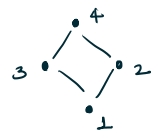$$\begin{cases} f([p_i, p_j]) & \text{if } p_i \leq p_j \\ 0 & \text{if } p_i \not\leq p_j \end{cases}$$

Example: $f([x,y]) = x+y$. Order $(1,2,3,4)$

$$\begin{array}{c} & 1 & 2 & 3 & 4 \\ 1 & \begin{bmatrix} 2 & 3 & 4 & 5 \\ 2 & 0 & 4 & 0 & 6 \\ 3 & 0 & 0 & 6 & 7 \\ 4 & 0 & 0 & 0 & 8 \end{bmatrix} \end{array} = M_f$$

↑
all these entries have to be zero because we had a topological sort.

Prop: If $M_f$ is the matrix of some $f \in A_P$ with respect to a topological sorting of $P$, then $M_f$ is always upper-triangular.

[If $i > j$ then $(M_f)_{(i,j)} = 0$.]

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$$g([x,y]) = x - y$$

$$M_g = \begin{array}{c} & 1 & 2 & 3 & 4 \\ 1 & \begin{bmatrix} 0 & -1 & -2 & -3 \\ 2 & 0 & 0 & 0 & -2 \\ 3 & 0 & 0 & 0 & -1 \\ 4 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Let's computing $M_f \cdot M_g$ (usual matrix product):

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 0 & 6 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & -4 & -16 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M_f \cdot M_g.$$

In fact, $M_f \cdot M_g = M_{(f * g)}$. [Check!]

Why? : The $(i,j)^{th}$ entry of $(M_f \cdot M_g)$

$$= \sum_{k=1}^{n} (M_f)_{(i,k)} (M_g)_{(k,j)}$$

Note: $(M_f)_{(i,k)} = \begin{cases} f([p_i, p_k]) & \text{if } p_i \leq p_k \\ 0 & \text{otherwise.} \end{cases}$

$$\longrightarrow \sum_{p_i \leq z \leq p_j} f([p_i, z]) \cdot g([z, p_j])$$

**Thm:** Let $(P, \leq)$ be a poset. Let $f, g \in A_P$.

Then

① $M_f \cdot M_g = M_{(f*g)}$     ← convolution

② $M_f + M_g = M_{(f+g)}$     ← addition

③ $r \cdot M_f = M_{rf}$    if   $r \in R$. ← scalar product.