MATH 2301 27 August 2020

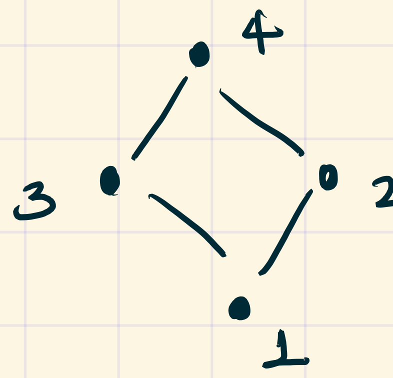
* Admin: Mid-term at 6:30 pm on 3^{rd} Sep.

Practice questions to be posted today. Typed answers will not be posted, but you are encouraged to discuss with me, each other, & demonstrators. Come to office hour!

* Last time: Addition + scalar multiplication in the incidence algebra

* Today: Convolution product.

E.g.



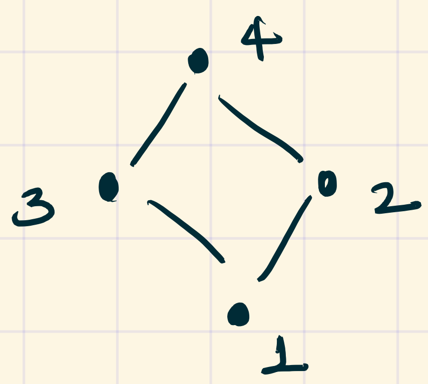
is $(P, \preceq)$. Recall: $I(P)$ = set of closed intervals of P.

[in this case, $I(P)$ has 9 elements]

Definition: Let $(P, \preceq)$ be a poset. Let $f, g \in \mathcal{A}_P$. Their convolution product $(f * g)$ is another element of $\mathcal{A}_P$, defined as follows: on any interval $[x, y]$

$$(f*g)([x,y]) := \sum_{x \preceq z \preceq y} f([x,z]) \cdot g([z,y])$$

Warning: $(f*g)([x,y]) \neq f([x,y]) \cdot g([x,y])$



Example: $f([x,y]) := x+y$

i.e. $f([1,3]) = 4$

$f([1,4]) = 5$

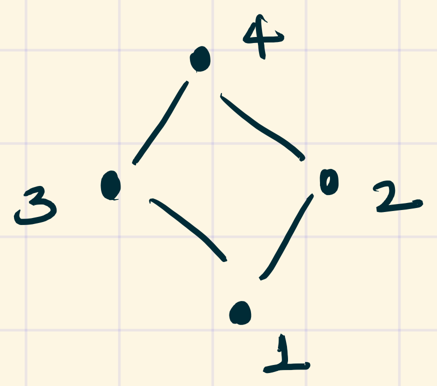
$g([x,y]) = x-y$.

$$(f*g)([1,1]) \stackrel{?}{=} \sum_{1 \preceq z \preceq 1} f([1,z]) \cdot g([z,1])$$

$z=1$ is the only option

$$= f([1,1]) \cdot g([1,1]) = (1+1)(1-1) = 0.$$

$$(f*g)([1,4]) = \sum_{1 \le z \le 4}' f([1,z])\, g([z,4])$$ there are 4 possibilities for z



$$= f([1,1]) \cdot g([1,4]) + f([1,2]) \cdot g([2,4])$$
$$+ f([1,3])\, g([3,4]) + f([1,4]) \cdot g([4,4])$$
$$= 2(-3) + 3(-2) + 4(-1) + 5(0)$$
$$= -6 + -6 + -4 = -16.$$

Exercise: Compute the value of $f*g$ on all 9 intervals.

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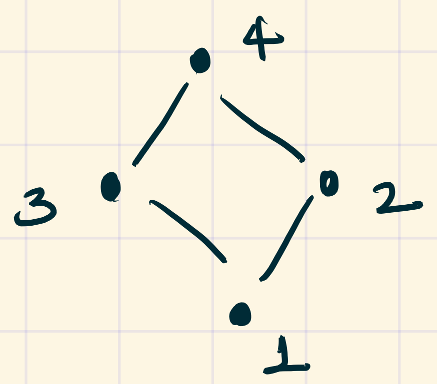
* Why this convolution product ??



① Order the vertices of your poset P.

Usually, we try to order "from bottom to top" so as to go with the flow of $\le$

$(1, 2, 3, 4)$, for example

But $(1, 3, 2, 4)$ is an equally good ordering with respect to this poset.

* These are nice orderings because whenever $a \le b$ in P, we have a appearing before b.

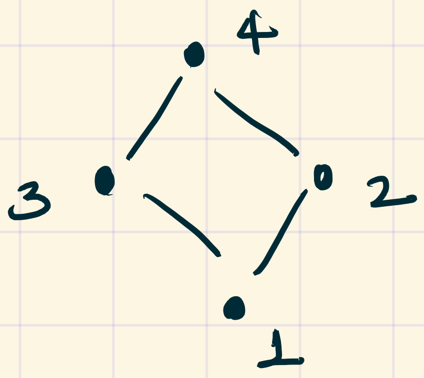
This is called a topological sorting of P.

Defn: Let $(P, \le)$ be a poset. An ordering $(p_1, \cdots, p_n)$ of the elements of P is called a topological sorting if whenever $p_i \le p_j$, we have $i \le j$.

* Warning: You might have $i < j$ with p_i & p_j incomparable.

* (Explanation of convolution product cont'd.)

① Consider a topological sort of P : $(p_1, \cdots, p_n)$



E.g $(1, 2, 3, 4)$

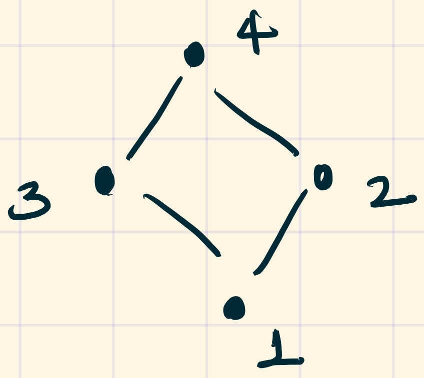
size of P

② Given any $f \in \mathcal{A}_P$, we'll write down an $n \times n$ matrix

M_f, defined as follows.

The $(i,j)^{th}$ entry of M_f is:

$$\begin{cases} f([p_i, p_j]) & \text{if } p_i \leq p_j \\ 0 & \text{if } p_i \not\leq p_j \end{cases}$$



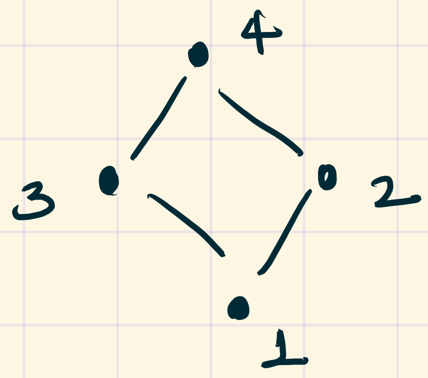
Example: $f([x,y]) = x+y$. Order $(1,2,3,4)$

$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 0 & 6 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix} \end{array} = M_f$$

all these entries have to be zero because we had a topological sort.

Prop: If M_f is the matrix of some $f \in \mathcal{A}_P$ with respect to a topological sorting of P, then M_f is always upper-triangular.

[If $i > j$ then $(M_f)_{(i,j)} = 0$.]



$$g([x,y]) = x-y$$

$$M_g = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \\ 2 & \\ 3 & \\ 4 & \end{array} \begin{bmatrix} 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let's computing $M_f \cdot M_g$ (usual matrix product):

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 0 & 6 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & -4 & -16 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M_f \cdot M_g.$$

In fact, $M_f \cdot M_g = M_{(f*g)}$.
[Check!]

Why? : The $(i,j)^{th}$ entry of $(M_f \cdot M_g)$

$$= \sum_{k=1}^{n} (M_f)_{(i,k)} (M_g)_{(k,j)}$$

Note: $(M_f)_{(i,k)} = \begin{cases} f([p_i, p_k]) & \text{if } p_i \preceq p_k \\ 0 & \text{otherwise.} \end{cases}$

$$\sum_{p_i \preceq z \preceq p_j} f([p_i, z]) \cdot g([z, p_j])$$

Thm: Let $(P, \preceq)$ be a poset. Let $f, g \in \mathcal{A}_P$.

Then ① $M_f \cdot M_g = M_{(f*g)}$. ← convolution

② $M_f + M_g = M_{(f+g)}$ ← addition

③ $r \cdot M_f = M_{rf}$ if $r \in \mathbb{R}$. ← scalar product.