MATH 2301   28 August 2020

* Admin : Midterm on 3^{rd} Sep, 6:30 pm – 8:30 pm
  Practice problems posted + info document posted.

* Current topic : The incidence algebra of a poset & its matrix representation.

* Rmk : In this section, we only consider finite posets.

* Recall : If $(P, \leq)$ a poset then $I(P)$ = set of closed intervals.

$$A_P = \{ f : I(P) \to \mathbb{R} \} + 3 \text{ operations.}$$

(incidence algebra)

- addition (pointwise)
- scalar multiplication
- multiplication (convolution)

Defn : Let $f, g \in A_P$.

$$(f * g)([x,y]) = \sum_{x \leq z \leq y}' f([x,z]) \cdot g([z,y])$$

* Matrix representation

① Fix a topological sorting of $P : (p_1, \dots, p_n)$ such that if $p_i \leq p_j$ then $i \leq j$

② For each $f \in A_P$, construct a matrix M_f

$$(M_f)_{(i,j)} = \begin{cases} f([p_i, p_j]) & \text{if } p_i \leq p_j \\ 0 & \text{otherwise.} \end{cases}$$

* Rmk : If M is any matrix such that $M_{ij} = 0$ whenever $p_i \not\leq p_j$, then it comes from some $f \in A_P$.
  In this case $f([p_i, p_j]) = M_{ij}$

③ In this case,

Theorem

$$M_{(f*g)} = M_f \cdot M_g$$

$\uparrow$ usual matrix product

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* Properties (Assuming a fixed topological sort)

- For any f, M_f is upper-triangular $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

- $f_0([x,y]) = 0 \ \forall$

  $\uparrow$ additive identity:
  $f_0 + g = g$ for every g

$$M_{f_0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix}$$

$M_{f_0} + M = M$ for any M.

- $\delta([x,y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

$$M_\delta = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \ 1 \end{bmatrix} = I_n$$

$\uparrow$ identity matrix.

$M_\delta \cdot M = M = M \cdot M_\delta$

multiplicative identity.

Claim: $f * \delta = \delta * f = f$ for every f.

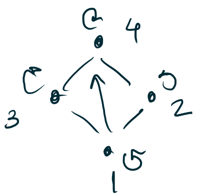
$$(f*\delta)([x,y]) = \sum_{x \leq z \leq y} f([x,z]) \cdot \underbrace{\delta([z,y])}_{= 0 \text{ unless } z = y}$$

$$= f([x,y]) \, \delta([y,y])$$

$$= f([x,y])$$

So, δ is the multiplicative identity for $*$.

$*\ \zeta([x,y]) = 1$ for every $x \preceq y$.



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_\zeta = \begin{bmatrix} \ddots & & \\ & \ddots & \\ 0 & & \ddots \end{bmatrix}$$

= adjacency matrix.

Q: Invertibility with respect to convolution?

→ we've noted that δ is the multiplicative identity.

Defn: Let $f \in \mathcal{A}_P$. We say that f is <u>invertible</u> if there is some $g \in \mathcal{A}_P$, such that $f * g = \delta$.

Rmk: If such a g exists, then actually, $g * f = \delta$ as well.

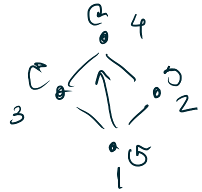
& similarly, if you find a g such that $g * f = \delta$, then automatically

Thm: An upper triangular $n \times n$ matrix is invertible if and only if every diagonal entry is non-zero. In this case, the inverse is also upper Δ^r.

* Note: Since M_ζ is the adjacency matrix and $\preceq$ is a (reflexive) partial order relation, M_ζ is an upper Δ matrix (by topological sort), and all diagonal entries are 1.

$\Rightarrow$ M_ζ always invertible.

⟷ this tells us, by going back to functions, that ζ has an inverse under convolution.



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A = M_\zeta$$

Let's try to find the inverse of A.

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & a+b & a+c & a+b+c+d \\ 0 & e & f & e+f+g \\ 0 & 0 & h & h+i \\ 0 & 0 & 0 & j \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1+b & 1+c & 1+b+c+d \\ 0 & 1 & f & 1+f+g \\ 0 & 0 & 1 & 1+i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ 0 & & & 1 \end{bmatrix}$$

$1+b = 0$, $f = 0$, $1+i = 0$

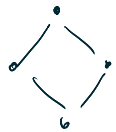
$b = -1$, $f = 0$, $i = -1$

$$\begin{bmatrix} 1 & 0 & 1+c & 1-1+c+d \\ 0 & 1 & 0 & 1+0+g \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ 0 & & & 1 \end{bmatrix}$$

$c = -1$, $g = -1$, $d = 1$

The inverse of A is:

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ & 1 & 0 & -1 \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} = A^{-1}$$



In fact, you can do this construction for any poset to produce the inverse for ζ.

This inverse is called μ, Möbius function of the poset $(P, \preceq)$.

In fact, μ has a recursive formula! $\to$ Next time