MATH 2301 02 September 2020

* Recap: We discussed multiplicative inverses in $\mathcal{A}_P$.

We say that $f \in \mathcal{A}_P$ is invertible if $\exists\, g \in \mathcal{A}_P$ such that $f * g = \delta$, or equivalently $g * f = \delta$. ← mult. identity

We recalled that: an $n \times n$ upper triangular matrix is invertible if and only if every diagonal entry is non-zero. The equivalent statement is true for $\mathcal{A}_P$:

Thm: $f \in \mathcal{A}_P$ is invertible iff $f([x,x]) \neq 0$ for every $x \in P$.

* Consequence: ζ is invertible!

We call this inverse μ = the Möbius function
(mu)

* Today's topic: the Möbius function $\mu \in \mathcal{A}_P$
(a recursive formula!)

Let's build up μ inductively "from the bottom up"

① First, for any $x \in P$, let's compute $\mu([x,x])$

* We heavily use the equation(s) $\mu * \zeta = \delta = \zeta * \mu$.

$$(\mu * \zeta)([x,x]) = \sum_{x \preceq z \preceq x} \mu([x,z]) \cdot \zeta([z,x])$$

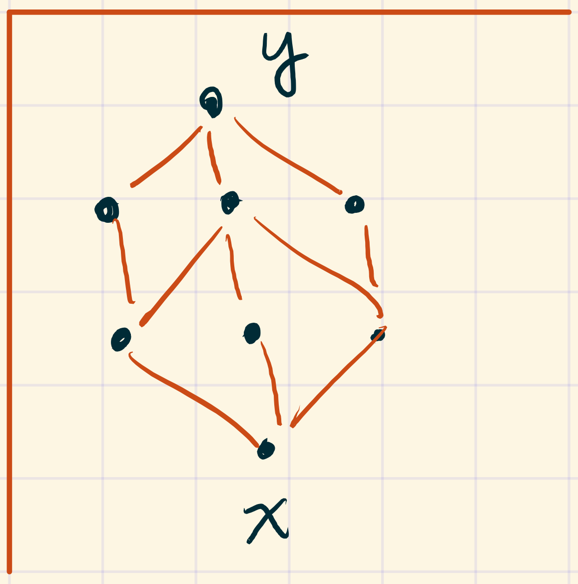
$z = x$ only option

$$(\mu * \zeta)([x,x]) = \delta([x,x]) = 1$$

$$\mu([x,z]) \cdot \zeta([z,x]) = \mu([x,x]) \cdot \zeta([x,x]) = \mu([x,x])$$

We see that $\mu([x,x]) = 1$ for every $x \in P$.

② Second, consider $x \precneqq y$. By induction, suppose we know the value of $\mu([x,z])$ for every $x \preceq z \precneqq y$.



Again, $(\mu * \zeta)([x,y]) = \delta([x,y]) = 0$

Also, $(\mu * \zeta)([x,y]) = \sum_{x \preceq z \preceq y} \mu([x,z])\, \underline{\zeta([z,y])}$ $\swarrow$ $[\zeta([z,y]) = 1$ for any $z \preceq y]$

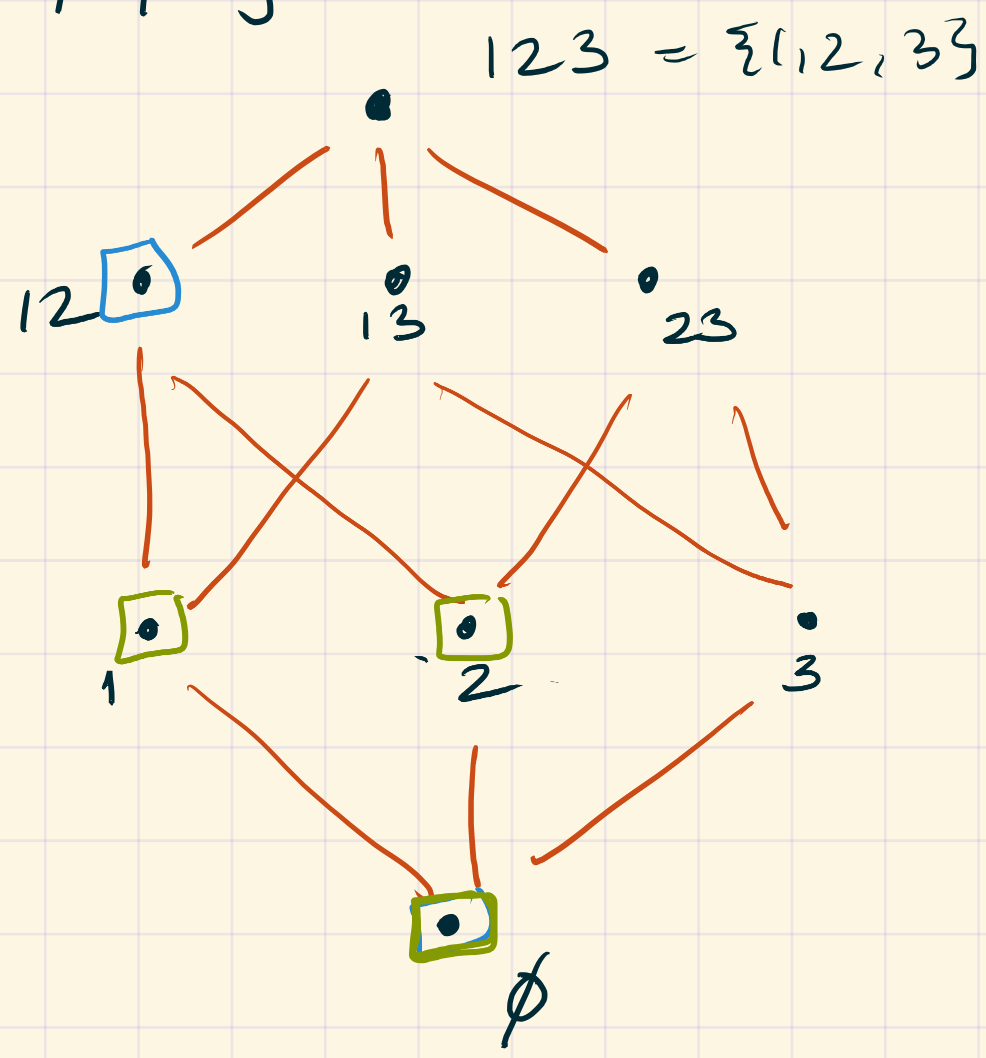
$$\| \qquad\qquad\qquad$$

$$0 = \sum_{x \preceq z \preceq y} \mu([x,z])$$

$$0 = \underbrace{\mu([x,y])}_{\text{isolate the term that we want}} + \underbrace{\sum_{x \preceq z \precneqq y} \mu([x,z])}_{\text{known by induction!}}$$

$$\Rightarrow \boxed{\mu([x,y]) = - \sum_{x \preceq z \precneqq y} \mu([x,z])}$$

Example: $P =$ subset poset of $\{1,2,3\}$



$\mu([\phi,\phi]) = 1$

$\mu([\phi, 12]) = ?$

$$\mu([\phi, 12]) = - \sum_{\phi \preceq z \precneqq \{1,2\}} \mu([\phi, z])$$

$$= -\mu([\phi,\phi]) - \mu([\phi, 1]) - \mu([\phi, 2])$$

$$= -1 - \underline{\mu([\phi, 1])} - \mu([\phi, 2]) \quad (*)$$

$$\mu([\phi, 1]) = - \sum_{\phi \preceq z \precneqq \{1\}} \mu([\phi, z]) = -\mu([\phi,\phi]) = -1$$

Similarly, $\mu([\phi, 2]) = -1$ $\leftarrow$ CHECK!

Back in $(*)$:

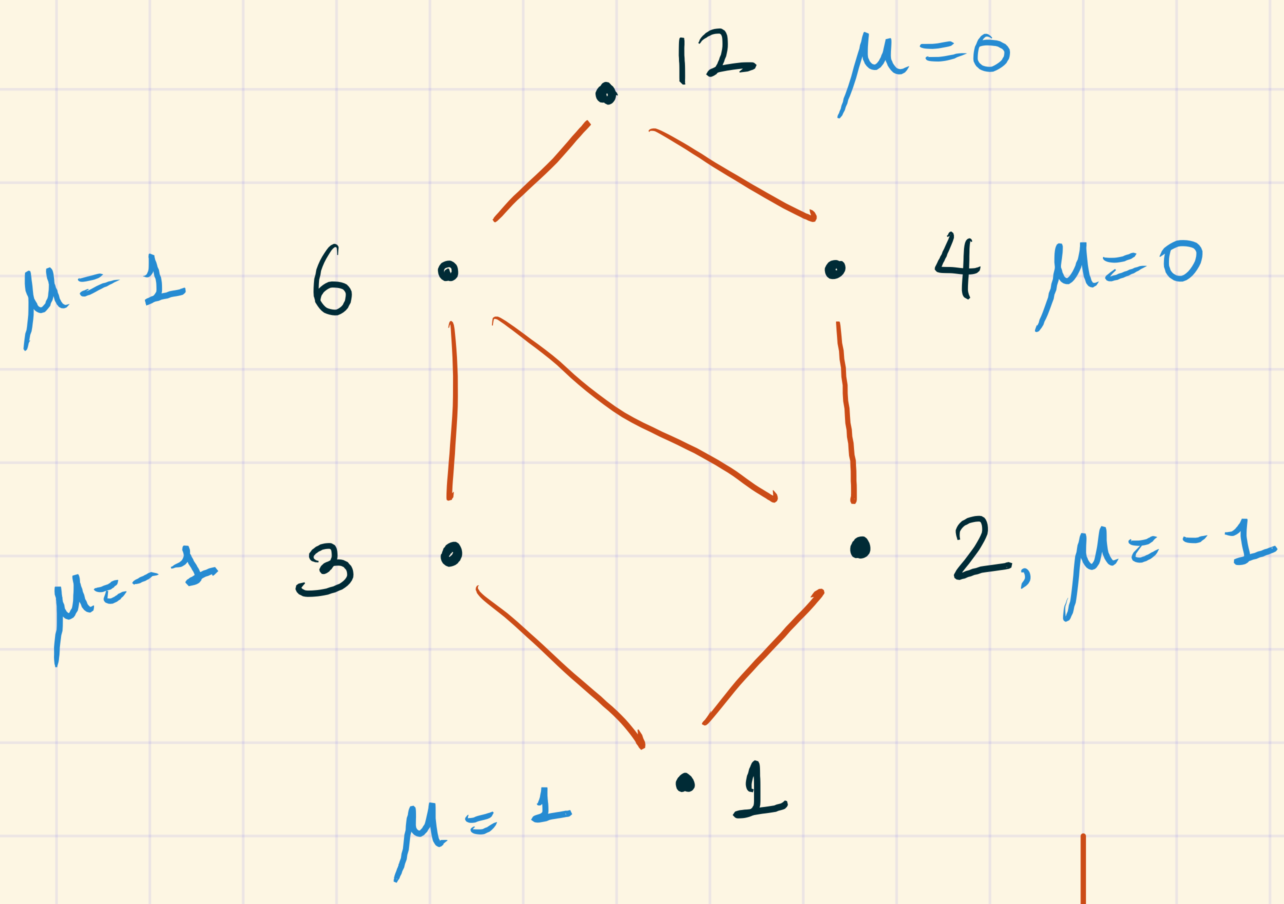
$$\mu([\phi, 12]) = -1 + 1 + 1 = 1.$$

Exercise: show: $\mu([\phi, 123]) = -1$

show: $\mu([1, 123]) = 1$

Challenge: Find a formula for $\mu([x,y])$ for a subset poset!

Classical example: Divisor poset [positive integer factors of some given n]



Q: $\mu([1,n])$? for $n \in P$

$\mu([1,1]) = 1$

$\mu([1,2]) = -\mu([1,1]) = -1$

$\mu([1,3]) = -1$

$\mu([1,6]) = -\mu([1,1]) - \mu([1,2]) - \mu([1,3])$

$= -1+1+1 = 1$

$\mu([1,4]) = -\mu([1,1]) - \mu([1,2])$

$= -1+1 = 0$

$\mu([1,12]) = -\mu([1,1]) - \mu([1,2]) - \mu([1,3]) - \mu([1,4]) - \mu([1,6])$

$= -1+1+1-0-1 = 0$

Defn: We say $n \in \mathbb{N}$ is square-free if there is no prime p such that n is divisible by p^2.

Eg: $1, 2, 3, 6$ are squarefree

$4, 12$ are not squarefree

Thm (Classical Möbius function): The value of $\mu([1,n])$ in the divisor poset for n equals:

① $$\mu([1,n]) = \begin{cases} 0 & \text{if } n \text{ is not square-free} \\ (-1)^k & \text{where } k = \text{number of distinct prime factors of } n. \end{cases}$$

and

② $\mu([a,b]) = \mu([1, b/a])$ ← in divisor poset, $a \preceq b$ iff $a \mid b$, so $b/a \in \mathbb{N}$.

Example: $n = 15 = 3 \times 5$ ← square-free w/ 2 prime factors

- $\mu([1,15]) = 1$
- $\mu([1,48]) = 0$

$\mu([1,30]) = (-1)^3 = -1$.

* Remarks

① $\mu([a,b])$ in some poset only depends on the shape (ie. the Hasse diagram) of the sub-poset between a & b. If the subposets $[a,b]$ & $[c,d]$ are isomorphic, then $\mu([a,b]) = \mu([c,d])$.

② For general posets, we don't have closed formulas for $\mu([a,b])$. But we have a general recursive formula, and we have closed formulas for the divisor poset, and for a few other "special" posets.
(not discussed.)

* Main application : Möbius inversion theorem (next time)
↳ chromatic polynomial